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Linear and Nonlinear Stability in non-standard Theories of Fluid Dynamics

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Introduction

In the last two centuries hydrodynamic stability has been recognized as one of the central problems of fluid mechanics. It is concerned with when and how laminar flows break down, their subsequent development, and their eventual transition to turbulence. It has many applications in engineering, in meteorology, in oceanography, in astrophysics and in geophysics. Today, the mathematical, computational and experimental aspects of the stability of laminar fluid motions have generated a huge and continuous flow of scientific researches. We refer to the book by Drazin and Reid [17] and references therein for the stability of hydrodynamic laminar flows and to the treatises by Chandrasekhar [11] and Straughan [87] for the problems related to the onset of convection in a viscous fluid in both hydrodynamic and hydromagnetic contexts.

In physical sciences one is first of all interested in linear stability of the models one applies to desribe a real world phenomenon. This is because in many situations linear stability analysis is mathematical feasible and gives the opportunity to catch the essential informations, but in some situations it is necessary to perform a nonlinear stability analysis as the it gives sufficient conditions for stability whereas the normal modes analysis furnishes only sufficient conditions for instability. The main method to study the nonlinear stability in hydrodynamics is certainly the Liapunov direct method. This method is introduced and explained in all the details in the book by Flavin and Rionero [20]. The core of this methodology is the choice of a suitable physically meaningful Liapunov functional such as the energy associated with the perturbations to a basic solution of the PDEs adopted to model a real world phenomenon. Although the subsequent analysis can be highly intricate, the Liapunov direct method is quite powerful and applicable in many situations. The aim of this thesis is to apply the Liapunov direct method to some implicit constitutive theories for fluids, in particular to fluids with temperature and pressure dependent viscosity.

In his celebrated paper on the constitutive response of fluids Stokes [85] recognized that the viscosity of a fluid could depend on the pressure although the experiments of Du Buat on the motion of water in canals and pipes at normal pressures showed that the total retardation of the velocity due to friction is not increased by increasing the pressure. This, however, does not seem to be the case at high pressures, even for incompressible liquids such as water: it is obvious that the viscosity of the water near the surface of the Pacific Ocean would be far less than viscosity near the bottom! In fact, it has long been known that the viscosity of typical liquids begins to increase sustantially with pressure when pressures on the order of 1000 atm are reached. (References to much of the experimental studies concerning the pressure dependence of viscosity can be found in [65].)

To have an idea of how viscosity changes with pressure we may refer to some of the most important experimental works on the subject. To take into account these experimental evidences Barus [5] proposed the following exponential dependence of viscosity μ on pressure p in isothermal processes:

$$\mu(p) = \mu_0 \exp[\beta(p - p_0)],$$

where μ_0 is the viscosity at the reference pressure p_0 and β is the piezo-viscous coefficient whose value varies with temperature. Later, Andrade [2] suggested a relationship among viscosity, pressure, density ρ and temperature T of the type:

$$\mu(\rho, p, T) = A\rho^{1/2} \exp\left[(p + \rho r^2)\frac{s}{T}\right]$$

where r, s and A are constants. More recently, Laun [36] modelled the viscosity of polymer melts as

$$\mu(p, T) = \mu_0 \exp[\beta(p - p_0) - \gamma(T - T_0)] \tag{1}$$

where μ_0 is the viscosity at reference pressure p_0 and temperature T_0 and the non-negative constants β and γ are, respectively, the pressure and temperature coefficient of viscosity. There have been numerous other experiments by Bair and co-workers that shows that the dependence of the viscosity on the pressure is exponential (see recent experiments of Bair and Kottke [4]). Mention must be made of the works of Martín-Alfonso and co-workers [46, 47] wherein an intricate relationship among the temperature, viscosity and pressure is provided for bitumen. In this context, it ought to be pointed out that the pressure dependence of the properties of bitumen were recognized very early. For instance, Saal and Koens [79] not only allowed for viscosity to depend on pressure (the normal stress), they even allowed it to depend on the shear stresses. Thus, they had a truly implicit constitutive model relating the stress and kinematical quantities (see also Saal and Labout [80] and Murali-Krishnan and Rajagopal [54]).

The theoretical results regarding the pressure-dependent viscosity fluids are, to our knowledge, still few and most of them are devoted to the determination of particular classes of flows (see, for instance, [30, 64, 70, 93]). The only results on the qualitative analysis of the equations governing the motion in a fluid with a pressure dependent viscosity, as far as we know, are due to Rajagopal and various co-workers [7, 44, 45] who have proved the existence of weak solutions for spatially periodic three-dimensional flows that are global in time for a large class of physically meaningful viscosity-pressure relationships. To our knowledge, nothing has been done in literature about the stability of flows in fluids with pressure dependent viscosity. In particular we found no result concerning the stability analysis of the Bénard problem for fluids with temperature and pressure dependent viscosity although it could be of practical interest in geophysics and in polymer melt processing. On the contrary, thermal convection for fluids with constant or temperature-dependent viscosity has been largely studied by many authors (see, for instance, [8, 9, 10, 11, 15, 17, 60, 86, 87, 92] and references therein).

A brief outline of the contents is now given. In Chapter 1 we introduce the basic concepts and notions on linear and nonlinear stability and illustrate the fundamental features of the Liapunov direct method. In particular, we introduce the energy method (which can be thought to be a particular case of the Liapunov direct method) we use to analyze the nonlinear stability of the Rayleigh-Bénard convection in a fluid with temperature and pressure dependent viscosity (Chapter 4), and of the laminar flows in an electrically conducting fluid saturating a porous medium (Chapter 5).

In Chapter 2 we deduce the governing equations of fluid mechanics by appealing to the implicit constitutive theories for fluids formulated by Rajagopal in [65]. We follow this innovative approach as we are taking into account the dependence on pressure of the fluid viscosity. Indeed the standard procedure in classical mechanics is to split the Cauchy stress tensor into the sum of two terms: the constraint stress and the so-called 'extra' stress. The former is assumed to not depend on the state variables (in the case of the classical fluid the velocity gradient) and, according to the Constraint Principle of Truesdell and Noll [90], does no work; the latter is constitutively prescribed but is assumed to not depend on the constrained part. Therefore, in the context of classical mechanics the fluid viscosity cannot depend on pressure and our choice of following Rajagopal [65] is then motivated by this lack in the standard theory of fluid dynamics.

When the dependence of viscosity on pressure is taken into account, the Oberbeck-Boussinesq equations, i.e. the approximate equations of motion of a heat-conducting viscous fluid under the action of gravity, must be slightly modified as one needs to distinguish between the pressure due to gravity and the pressure due to the thermal expansion of the fluid, only the former contributes to variations in viscosity at a first approximation. The first original step in this thesis is then to provide a rigorous mathematical justification for the Oberbeck-Boussinesq approximation when all the material parameters of the fluid (the viscosity μ, the thermal conductivity k, the specific heat at constant pressure c_p and the coefficient of volumetric thermal expansion α) are analytic functions of pressure and temperature, and to derive approximate equations governing the motion of a heat-conducting viscous fluid (see also [67]). The approximate governing equations we obtain reduce to the classical Oberbeck-Boussinesq equations if the material parameters of the fluid are not dependent on pressure.

While it is true that all the physical quantities do vary with pressure, the variation in the viscosity with pressure is far more dramatic than the variation of the other quantities with pressure. For instance, while viscosity might change by a factor of ten to the power of eight or more (see [4]), the density will vary by merely a few percent (see [16], [62] for details). The other properties also undergo much more modest changes in their values than the viscosity and hence we feel that assuming α, k and c_p constants is a reasonable first approximation. Based on this approximation, we both determine the laminar flows (Chapter 3) and study the onset of convection (Chapter 4) in fluids whose viscosity depends analytically on both the temperature and pressure.

In Chapter 3, by taking into account the viscosity model proposed by Laun for polymer melts, we observe how the pressure-temperature dependent viscosity influences the shape of the velocity profiles in parallel shear flows. Moreover we see that the velocity profiles in Poiseuille and Couette flows in bitumen differ not so much from the classical case in which viscosity is assumed to be a constant in spite of the intricate bitumen viscosity model proposed by Martin-Alfonso and co-workers [46, 47]. Indeed here we consider what happens in a laboratory while in geophysical field applications the effect of the pressure dependence may be more dramatic.

As concerns the onset of convection in a pressure-temperature dependent viscosity fluid, in Chapter 4 we report the results in [68] in which we prove that the principle of exchange of stabilities holds and hence instability sets in as stationary convection. Then, by following a standard procedure, we show how to find the critical Rayleigh number, the linear stability/instability threshold in Bénard problem, by appealing to a variational analysis. The nonlinear energy stability analysis yields that the thresholds for the linear theory and energy analysis coincide provided the initial disturbance to the temperature field meets a specific restriction. We may then state that the basic results of the classical theory (validity of the principle of exchange of stabilities and coincidence of the linear instability and energy stability thresholds) are extended to fluids whose viscosity depends analytically on both temperature and pressure. Moreover we present approximations to the critical Rayleigh number both for rigid and stress-free boundary conditions when the viscosity depends on pressure and temperature as in (1). These approximations are obtained by employing the Galerkin-type method developed by Chandrasekhar in [11] whose convergence is discussed in Section 5.5 in relation to the characteristic-value problem raised by the stability analysis of the laminar flows studied in Chapter 5.

Another problem involving the stability of the shear parallel flows we discuss in this thesis is the stability of the laminar motions in an electrically conducting fluid saturating a porous medium. It is known that in the geothermal region the sub-surface ground water posseses a general upward convective drift due to buoyancy induced by the high underground temperature. Since the rising ground water is cooled as it approaches the surface, where heat is removed by evaporation, radiation and movement in the surface streams, an unstable state may be induced and complicated convective motions appear in the layers near the surface. In those circumstances it is of practical interest to consider the effect of the geomagnetic field on such flows and see whether the magnetic field inhibits this instability. In [78] Rudraiah and Mariyappa studied the stability of steady hydromagnetic flows in a porous medium by assuming the fluid with a finite electrical conductivity, valid the Oberbeck-Boussinesq approximation and neglecting the effects of its viscosity with respect to the friction that manifests itself at the pores. In Chapter 5, instead, we include the frictional forces in the fluid by considering the unsteady Brinkman model for flows of a viscous fluid in a porous medium, determine the laminar flows, show how both the magnetic field and the porous matrix influence the shape of the velocity profiles and find sufficient conditions for linear and nonlinear energy stability (see also [75]).

Chapter 1

Basic concepts and mathematical methods

1.1 Evolution equations. Dynamical systems

Let $\mathcal{F}$ be a phenomenon occurring in a domain Ω of the physical three-dimensional space $\mathbb{R}^3$ and let $v_i(\mathbf{x}, t)$ - with $i = 1, 2, ..., n$ $(n \in \mathbb{N})$, $\mathbf{x} \in \Omega$ and $t \in \mathbb{R}^+$ an instant of time - represent the relevant quantities describing the state of $\mathcal{F}$. The vector $\mathbf{v} \equiv (v_1, v_2, ..., v_n)$ is the *state vector*. The phenomenon $\mathcal{F}$ is modelled by a P.D.E. if one can establish the existence of a function

$$\mathbf{F}\left(\mathbf{x}, t, \mathbf{v}, \frac{\partial v_i}{\partial x_r}, \frac{\partial^2 v_j}{\partial x_r \partial x_s}, ...\right), \quad i, j = 1, 2, ..., n; \ r, s = 1, 2, 3$$

which governs the behaviour of the time derivative of $\mathbf{v}$, viz, for any $T > 0$,

$$\mathbf{v}_t = \mathbf{F} \qquad \text{in } \ \Omega \times (0, T) \tag{1.1}$$

subject to the initial data

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \Omega \tag{1.2}$$

and suitable boundary conditions

$$A(\mathbf{v}, \nabla\mathbf{v}) = \hat{\mathbf{v}} \quad \text{on } \partial\Omega \times (0, T) \tag{1.3}$$

where $\mathbf{v}_0(\mathbf{x}, t)$ and $\hat{\mathbf{v}}(\mathbf{x}, t)$ are prescribed functions and A is an assigned operator.

The initial-boundary value problem (I.B.V.P.) (1.1)-(1.3) is a mathematical model for the evolution of the state vector $\mathbf{v}$ of the phenomenon $\mathcal{F}$ and therefore it represents the *evolution equation* of $\mathcal{F}$. The space X of vector valued functions defined in Ω and satisfying the prescribed boundary conditions, endowed with an appropriate metric, is called the *state space*. The choice of the metric is the core of the problem and has to be linked to the physics of the phenomenon (see [20] for a detailed discussion).

Let X be the state space of the evolution equation (1.1) endowed with a metric d suitably chosen. As a first indication that the model of the real world phenomenon $\mathcal{F}$ is correct, one requires the *well posedness* in the sense of Hadamard [97]. Hadamard's conditions for a well posed problem are:

i) the existence of a solution;

ii) the uniqueness of the solution;

iii) the continuous dependence of the solution on the data.

The first two conditions require that the I.B.V.P. (1.1)-(1.3) admits one and only one global (in time) solution, that is the solution exists for every finite interval of time. The third condition states that a slight variation of the data for the problem should cause the solution to vary only slightly. Thus since data are generally obtained experimentally and may be subject to numerical approximations, one requires that the solution be stable under small variations in initial and/or boundary data. We shall now make this last requirement clearer by formalizing it in a mathematically rigorous way.

Let $\mathbf{v}(\mathbf{v}_0, t)$ be a global solution to the problem (1.1)-(1.3). $\mathbf{v}$ is a dynamical system according to the following definition [34, 94]

Definition 1.1. *A dynamical system on a metric space X is a map*

$$\mathbf{v} : (\mathbf{v}_0, t) \in X \times \mathbb{R} \mapsto \mathbf{v}(\mathbf{v}_0, t) \in X$$

such that $\mathbf{v}(\mathbf{v}_0, 0) = \mathbf{v}_0$.

Given $\mathbf{v}_0 \in X$, for a dynamical system $\mathbf{v}$, the function

$$\mathbf{v}(\mathbf{v}_0, \cdot) : t \in \mathbb{R} \mapsto \mathbf{v}(\mathbf{v}_0, t) \in X$$

is called a *motion* associated with the initial data $\mathbf{v}_0$. If

$$\mathbf{v}(t) = \mathbf{v}_0 \quad \forall t \in \mathbb{R}^+,$$

the motion is *stationary* or *steady* and $\mathbf{v}_0$ is an *equilibrium* or *critical point*. The set $\{(t, \mathbf{v}(t)) : t \in \mathbb{R}^+\}$ is the *positive graph* of the motion $\mathbf{v}$, and its projection into X, i.e. the subset $\gamma(\mathbf{v}_0) = \{\mathbf{v}(t) : t \in \mathbb{R}^+\}$, is the *positive orbit* or the *trajectory* starting at $\mathbf{v}_0$. A subset $I \subset X$ is *positively invariant* if $\mathbf{v}_0 \in I \Rightarrow \gamma(\mathbf{v}_0) \subset I$. If there exists $T > 0$ such that $\mathbf{v}(t + T) = \mathbf{v}(t)$ $\forall t \in \mathbb{R}$, the motion $\mathbf{v}$ is *periodic in time with period T*.

Returning to the mathematical formalization of the reqirement *iii*) and denoting by $S(\mathbf{x}, r)$ $(r > 0)$ the open ball in the metric space (X, d) of centre $\mathbf{x}$ and radius r,

$$S(\mathbf{x}, r) = \{y \in X : d(\mathbf{x}, \mathbf{y}) < r\},$$

we state the following

Definition 1.2. *A motion* $\mathbf{v}(\mathbf{v}_0, \cdot)$ *depends contionuously on the initial data if and only if*

$$\forall T > 0,\ \forall \epsilon > 0\ \exists \delta(\epsilon, T) > 0\ :\ \mathbf{v}_1 \in S(\mathbf{v}_0, \delta) \Rightarrow \\ \mathbf{v}(\mathbf{v}_1, t) \in S(\mathbf{v}(\mathbf{v}_0, t), \epsilon)\ \forall t \in [0, T]. \tag{1.4}$$

1.2 Ill-posed problems

A problem which is not well posed is said to be *ill posed*. Ill posedness is then due to the lack of one of the requirements *i*), *ii*), *iii*) in the previous section.

Example 1.1 (Lack of existence)**.** Let

$$\sum_{|\alpha| \le k} a_\alpha(x_1, \ldots, x_n) \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = F(x_1, \ldots, x_n), \tag{1.5}$$

be a partial differential equation in which $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ is a multi-index of length $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $k \in \mathbb{N}$, the coefficients a_α and the datum F are assigned analytic functions of the real variables $x_1, \ldots, x_n$. Assume that there exists $\mathbf{x}_0 \in \mathbb{R}^n$ at which at least one of the functions a_α, with $|\alpha| = k$, does not vanish. Then, the Cauchy-Kovalewski Theorem ensures the existence of a solution to (1.5) in a neighborhood of $\mathbf{x}_0$. If we weaken the hypothesis of analyticity of the datum by assuming $F \in \mathrm{C}^\infty(\mathbb{R}^n)$, the existence of a classical solution is not guaranteed. To this end we report the celebrated Lewy's example [37]

$$u_x + \mathrm{i} u_y - 2\mathrm{i}(x + \mathrm{i} y) u_t = F(x, y, t) \quad (x, y, t) \in \mathbb{R}^3, \tag{1.6}$$

where i is the imaginary unit. Although the coefficients in equation (1.6) are all analytic functions, Lewy was able to show that there exists $F \in \mathrm{C}^\infty(\mathbb{R}^3)$ such that (1.6) has no C^1 solution anywhere in $\mathbb{R}^3$. Therefore, by adding to (1.6) an initial condition

$$u(x, y, 0) = u_0(x, y) \in \mathrm{C}^1(\mathbb{R}^2),$$

we have an example of ill posed I.V.P. in the state space $\mathrm{C}^1(\mathbb{R}^2)$ as it does not admit a classical solution.

Lack of uniqueness implies ill posedness also because it guarantees that continuous dependence cannot be obtained. In fact, if $\mathbf{v}$ is a dynamical system on a metric space X according to Definition 1.1, the following Theorem holds.

Theorem 1.1. *A motion which is not unique cannot depend contionuously on the initial data.*

Proof. Let $\mathbf{v}$ and $\mathbf{w}$ be two motions associated with the same initial data, i.e. $\mathbf{v}(0) = \mathbf{w}(0)$, such that there exists $t^* > 0$ for which $d(\mathbf{v}(t^*), \mathbf{w}(t^*)) = \epsilon^* > 0$. Then, for $T > t^*$ and $0 < \epsilon < \epsilon^*$, (1.4) does not hold. $\square$

Example 1.2 (Lack of uniqueness)**.** We shall now present a counterexample to uniqueness in fluid mechanics. The Navier-Stokes equations

$$\begin{cases} \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v} + \mathbf{b} \\ \\ \operatorname{div} \mathbf{v} = 0 \end{cases} \qquad (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \tag{1.7}$$

are a mathematical model describing the motion of an incompressible homogeneous viscous fluid occurring in a fixed region $\Omega \subseteq \mathbb{R}^3$. In (1.7) $\mathbf{x} = (x, y, z) \in \Omega$ is the space variable, $t \in \mathbb{R}^+$ the time, $\mathbf{v}$ the velocity field, p the pressure field divided by the constant density of the fluid, $\nu(> 0)$ the coefficient of kinematic viscosity and $\mathbf{b}$ the body force acting on the fluid. When the fluid adheres completely to the boundary $\partial\Omega$, the initial-boundary condition to append to (1.7) are:

$$\begin{cases} \mathbf{v}(\mathbf{x}, t) = \mathbf{v}_0(\mathbf{x}) & \mathbf{x} \in \Omega, \\ \mathbf{v}(\mathbf{x}, t) = \mathbf{v}^*(\mathbf{x}, t) & (\mathbf{x}, t) \in \partial\Omega \times \mathbb{R}^+, \end{cases} \tag{1.8}$$

where $\mathbf{v}_0$ and $\mathbf{v}^*$ are prescribed vector functions, $\mathbf{v}_0$ being divergence-free. Let us consider the case

$$\Omega \equiv \mathbb{R}^3, \quad \mathbf{b} = \mathbf{0} \quad \text{and} \quad \mathbf{v}_0 = \mathbf{0}.$$

Then one readily obtains that the I.V.P. (1.7)-$(1.8)_1$ (there is no boundary when $\Omega \equiv \mathbb{R}^3$) admits at least the following three solutions (see [22, 55] for other solutions):

$$\begin{aligned} &\mathbf{v} = \mathbf{0}, && p = p_0(t); \\ &\mathbf{v} = t(y\mathbf{i} + x\mathbf{j}), && p = p_1(t) - xy - \frac{x^2 + y^2}{2} t^2; \\ &\mathbf{v} = \sin t\mathbf{i} + \sinh t(z\mathbf{j} + y\mathbf{k}), && p = p_2(t) - x\cos t - \frac{y^2 + z^2}{2} \sinh^2 t - yz\cosh t; \end{aligned}$$

where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along the x, y and z axes, respectively, and $p_i(t)$ $(i = 0, 1, 2)$ are arbitrary functions.

Example 1.3 (Lack of contionuous dependence)**.** Let us consider the *backward heat equation*

$$\begin{cases} u_t = u_{xx} & x \in \mathbb{R},\ t < 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R}, \end{cases} \tag{1.9}$$

where u_0 is a prescribed $\mathrm{C}^2(\mathbb{R})$ function. Then we take $\mathrm{C}^2(\mathbb{R})$ endowed with the L^∞-norm,

$$\|f\|_\infty = \max_{x\in\mathbb{R}} |f(x)|,$$

as the state space. For $u_0 \equiv 0$, (1.9) admits the zero solution which, as we shall soon show, does not depend continuously on the initial datum. In fact, let

$$u_0 = u_{0n} = \frac{1}{n}\sin(nx) \quad (n \in \mathbb{N});$$

then

$$u_n = \frac{\mathrm{e}^{-n^2 t}}{n}\sin(nx)$$

is the solution to (1.9) and, for all $n \in \mathbb{N}$,

$$\|u_{0n}\|_\infty = \frac{1}{n}, \quad \|u_n\|_\infty = \frac{\mathrm{e}^{n^2|t|}}{n} > 1 \ \forall |t| > \frac{1}{2\mathrm{e}}.$$

Therefore, for $\epsilon = 1$ and $T > (2\mathrm{e})^{-1}$, (1.4) does not hold.

1.3 Liapunov stability

The Liapunov stability of a motion $\mathbf{v}(\mathbf{v}_0,\cdot)$ of a dynamical system $\mathbf{v}$ extends the requirements of continuous dependence to the infinite interval of time $(0,+\infty)$.

Definition 1.3. *A motion $\mathbf{v}(\mathbf{v}_0,\cdot)$ is stable in the sense of Liapunov with respect to perturbations in the initial data if and only if*

$$\begin{aligned} &\forall\epsilon > 0,\ \exists\delta(\epsilon) > 0 \ : \ \mathbf{v}_1 \in S(\mathbf{v}_0,\delta) \Rightarrow \\ &\qquad\qquad \mathbf{v}(\mathbf{v}_1,t) \in S(\mathbf{v}(\mathbf{v}_0,t),\epsilon) \quad \forall t \in \mathbb{R}^+. \end{aligned} \tag{1.10}$$

A motion is *unstable* if it is not stable. Obviously (1.10) implies (1.4) and hence, by means of Theorem 1.1, a motion which is stable is also unique.

Definition 1.4. *A motion $\mathbf{v}(\mathbf{v}_0,\cdot)$ is said to be an attractor (or attractive) on a set $Y \subseteq X$ if:*

$$\mathbf{v}_1 \in Y \Rightarrow \lim_{t\to+\infty} d[\mathbf{v}(\mathbf{v}_0,t),\mathbf{v}(\mathbf{v}_1,t)] = 0. \tag{1.11}$$

The biggest set Y on which (1.11) *holds is called the domain of attraction of* $\mathbf{v}(\mathbf{v}_0,\cdot)$.

Definition 1.5. *A motion* $\mathbf{v}(\mathbf{v}_0, \cdot)$ *is asymptotically stable if it is stable and there exists* $\delta > 0$ *such that* $\mathbf{v}(\mathbf{v}_0, \cdot)$ *is attractive on* $S(\mathbf{v}_0, \delta)$*. In particular* $\mathbf{v}(\mathbf{v}_0, \cdot)$ *is exponentially stable if*

$$\begin{aligned}\exists \delta,\ \lambda(\delta),\ M(\delta) \in \mathbb{R}^+ \ : \ \forall \mathbf{v}_1 \in X,\ d(\mathbf{v}_1, \mathbf{v}_0) < \delta \Rightarrow \\ d[\mathbf{v}(\mathbf{v}_1, t), \mathbf{v}(\mathbf{v}_0, t)] \leq M \mathrm{e}^{-\lambda t} d(\mathbf{v}_1, \mathbf{v}_0) \quad \forall t \in \mathbb{R}^+.\end{aligned}$$

If $\delta = +\infty$*, then* $\mathbf{v}(\mathbf{v}_0, \cdot)$ *is globally asymptotically (or exponentially) stable.*

Liapunov stability of a set is of fundamental interest expecially in connection with the asymptotic behaviour of motions. In order to formalize this notion we recall the definition of distance between two subsets A, B of a metric space X:

$$d(A, B) = \inf_{x \in A,\, y \in B} d(x, y) \tag{1.12}$$

and denote by $S(A, r)$, $r > 0$, the open set $\{x \in X : d(x, A) < r\}$, where $d(x, A) = d(\{x\}, A)$ according to (1.12).

Definition 1.6. *A set* $A \subset X$ *is Liapunov stable with respect to the dynamical system* $\mathbf{v}$ *if*

$$\forall \epsilon > 0 \,\exists\, \delta(\epsilon) > 0 \ : \ \mathbf{v}_0 \in S(A, \delta) \Rightarrow \gamma(\mathbf{v}_0) \subset S(A, \epsilon).$$

A set is *unstable* if it is not stable.

Definition 1.7. *A set* A *is said to be an attractor or attractive on an open set* $B \supset A$ *with respect to the dynamical system* $\mathbf{v}$ *if it is positive invariant and*

$$\mathbf{v}_0 \in B \Rightarrow \lim_{t \to +\infty} d[\mathbf{v}(\mathbf{v}_0, t), A] = 0. \tag{1.13}$$

The largest open set on which (1.13) *holds is the domain of attraction of* A*. If* $B = X$*, then* A *is a global attractor.*

Definition 1.8. *A set* A *is asymptotically stable if it is stable and if there exists* $\delta > 0$ *such that* A *is attractive on* $S(A, \delta)$*. In particular* A *is exponentially stable if*

$$\exists \delta, \lambda(\delta), M(\delta) \in \mathbb{R}^+ : \mathbf{v}_0 \in S(A, \delta) \Rightarrow d[\mathbf{v}(\mathbf{v}_0, t), A] \leq M \mathrm{e}^{-\lambda t} d(\mathbf{v}_0, A) \forall t > 0.$$

If the domain of attraction is the whole space X*, i.e.* $\delta = +\infty$*, then the asymptotic (or exponential) stability is said to be global.*

Remark 1.1. Let X be a normed linear space, $d : (x, y) \in X \times X \mapsto \|x - y\|$ the metric induced by the norm $\|\cdot\|$, $\mathbf{v}$ a dynamical system on X and

$$\mathbf{u}(\mathbf{u}_0, t) = \mathbf{v}(\mathbf{v}_1, t) - \mathbf{v}(\mathbf{v}_0, t) \quad (\mathbf{v}_1 = \mathbf{u}_0 + \mathbf{v}_0)$$

the perturbation at time t to the basic motion $\mathbf{v}(\mathbf{v}_0, \cdot)$. Then (1.10) is equivalent to

$$\forall \epsilon > 0 \ \exists \delta(\epsilon) > 0 \ : \ \|\mathbf{u}_0\| < \delta \Rightarrow \ \|\mathbf{u}(\mathbf{u}_0, t)\| < \epsilon \ \forall t \in \mathbb{R}^+,$$

viz the stability of a given basic motion $\mathbf{v}(\mathbf{v}_0, \cdot)$ may be expressed through the stability of the zero solution of the perturbed dynamical system

$$\mathbf{u} : (\mathbf{u}_0, t) \in X \times \mathbb{R}^+ \mapsto \mathbf{v}(\mathbf{v}_0 + \mathbf{u}_0, t) - \mathbf{v}(\mathbf{v}_0, t).$$

Remark 1.2. On a set X, a functional

$$\rho : X \times X \to \mathbb{R}$$

is a positive-definite function if it satisfies

a) $\rho(x, y) \geq 0 \quad \forall x, y \in X,$

b) $\rho(x, y) = 0 \Leftrightarrow x = y.$

A metric is obviously a positive-definite function but the converse is true when additionally there holds

1) $\rho(x, y) = \rho(y, x) \quad \forall x, y \in X,$

2) $\rho(x, y) \leq \rho(x, z) + \rho(z, y) \quad \forall x, y, z \in X.$

Furthermore we note that a positive-definite function does not define a topology. Nevertheless we define *open ball* in X with centre x and radius $r(> 0)$ the set

$$S_\rho(x, r) = \{y \in X : \rho(x, y) < r\}.$$

Sometimes a positive-definite function is chosen as a measure of the perturbations [35].

1.4 Topology dependent stability

In the applications the state space X is often a normed linear space $(X, \|\cdot\|)$ and a metric d is induced by the norm $\|\cdot\|$ as in Remark 1.1. It is well known that two equivalent norms induce two equivalent metrics and then the same topology [88]. Therefore stability (resp. instability) with respect to a fixed norm implies stability (resp. instability) with respect to an equivalent one. But only on a linear finite dimensional space all norms are equivalent [40] and consequently stability does not depend on the chosen norm. On an infinite dimensional space instead, it can turn out that a solution is stable with a choice of the metric and unstable with another one.

Example 1.4 (Fichera [19])**.** Let us consider the Cauchy problem

$$\begin{cases} u_t = \left(\frac{2}{t} - 6t^5x^2\right) u(x,t) & (x,t) \in [-1,1] \times [1,+\infty[\\ u(x,1) = f(x) & x \in [-1,1] \end{cases} \tag{1.14}$$

which, for $f \equiv 0$, admits the trivial solution $u \equiv 0$. By taking $\mathrm{C}^0[-1,1]$ as the state space and considering on it both the $\mathrm{L}^1[-1,1]$-norm,

$$\|w\|_1 = \int_{-1}^{1} |w(x)| \mathrm{d}x,$$

and the $\mathrm{L}^\infty[-1,1]$-norm, we shall show that the zero solution is stable with respect to the $\mathrm{L}^1[-1,1]$-norm and unstable with respect to the $\mathrm{L}^\infty[-1,1]$-norm. It is easy to check that

$$u(x,t) = f(x) t^2 \mathrm{e}^{-(t^6-1)x^2} \tag{1.15}$$

is the solution to (1.14) and thus

$$\begin{aligned} \|u(\cdot,t)\|_1 &= \int_{-1}^{1} |u(x,t)| \mathrm{d}x = \int_{-1}^{1} |f(x)| t^2 \mathrm{e}^{-(t^6-1)x^2} \mathrm{d}x \\ &\le \|f\|_\infty t^2 \int_{-1}^{1} \mathrm{e}^{-(t^6-1)x^2} \mathrm{d}x = \|f\|_\infty \frac{2t^2}{\sqrt{t^6-1}} \int_0^{\sqrt{t^6-1}} \mathrm{e}^{-\xi^2} \mathrm{d}\xi \\ &< \|f\|_\infty \frac{t^2}{\sqrt{t^6-1}} \int_{-\infty}^{+\infty} \mathrm{e}^{-\xi^2} \mathrm{d}\xi = \|f\|_\infty t^2 \sqrt{\frac{\pi}{t^6-1}} \to 0 \quad \text{as } t \to +\infty \end{aligned}$$

which implies the stability of the zero solution with respect to the $\mathrm{L}^1[-1,1]$-norm. On the other hand, if we choose as perturbation to the initial datum

$$f(x) = f_n(x) = \frac{\mathrm{e}^{-x^2}}{n} \quad (n \in \mathbb{N}),$$

the solution (1.15) to (1.14) becomes

$$u_n(x,t) = \frac{t^2}{n} \mathrm{e}^{-t^6x^2}.$$

Because of

$$\|f_n\|_\infty = \frac{1}{n}$$

the data tend to zero as $n \to +\infty$, while

$$\|u_n\|_\infty = \max_{x \in [-1,1]} \left| \frac{t^2}{n} \mathrm{e}^{-t^6x^2} \right| = \frac{t^2}{n} \to +\infty \quad \text{as } t \to +\infty \quad \forall n \in \mathbb{N},$$

by which the instability of the zero solution with respect to the $\mathrm{L}^\infty[-1,1]$-norm follows.

Example 1.5 (Hadamard)**.** We consider the initial value problem for the Laplace equation

$$\begin{cases} u_{tt} + u_{xx} = 0 & x \in [0,1],\ t > 0, \\ u(0,t) = u(1,t) = 0 & t > 0, \\ u(x,0) = 0, \quad u_t(x,0) = u^*(x), \end{cases} \tag{1.16}$$

with u^* a prescribed $\mathrm{C}^2[0,1]$ function vanishing at $x=0$ and $x=1$, and we take the linear space

$$X = \{f \in \mathrm{C}^2[0,1] : f(0) = f(1) = 0\},$$

endowed with the $\mathrm{L}^\infty[0,1]$-norm, as the state space.

For $u^* = 0$ the I.V.P. (1.16) admits the trivial solution. This solution is unstable with respect to the $\mathrm{L}^\infty[0,1]$-norm. In fact, by choosing

$$u_n^* = \frac{1}{n}\sin(n\pi x) \quad (n \in \mathbb{N})$$

as perturbation to the initial datum, the explicit solution to (1.16) is

$$u_n(x,t) = \frac{1}{n^2\pi}\sinh(n\pi t)\sin(n\pi x).$$

Thus, since

$$\|u_n^*\|_\infty = \frac{1}{n}$$

and

$$\|u_n\|_\infty = \frac{\sinh(n\pi t)}{n^2\pi},$$

it follows that, as $n \to +\infty$, the data tend uniformly to zero while $\|u_n\|_\infty$ tends to $+\infty$.

Recalling that the set of functions $\{\sin(k\pi x)\}_{k\in\mathbb{N}}$ is complete in the state space X under the $\mathrm{L}^\infty[0,1]$-norm, let $u^* \in X$,

$$u^*(x) = \sum_{k=1}^{+\infty} a_k \sin(k\pi x) \quad \text{with} \quad a_k = 2\int_0^1 u^*(x)\sin(k\pi x)\mathrm{d}x.$$

Then

$$u(x,t) = \sum_{k=1}^{+\infty} \frac{a_k}{k\pi}\sinh(k\pi t)\sin(k\pi x) \tag{1.17}$$

is the explicit solution to (1.16). Furthermore, we observe that, along the solutions (1.17), one has

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2 \mathrm{d}x = -\int_0^1 u_t u_{xx}\mathrm{d}x = -[u_t u_x]_0^1 + \int_0^1 u_x u_{tx}\mathrm{d}x,$$

and hence, since u_t vanishes at $x = 0$ and $x = 1$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(u_t^2 - u_x^2\right) \mathrm{d}x = 0,$$

by which

$$\int_0^1 \left(u_t^2 - u_x^2\right) \mathrm{d}x = \text{constant} = \int_0^1 \left(u_t^2 - u_x^2\right)_{t=0} \mathrm{d}x = \int_0^1 u^{*2}(x)\mathrm{d}x.$$

Therefore on the class of solutions (1.17), $1/2 \int_0^1 \left(u_t^2 - u_x^2\right) \mathrm{d}x$ can be assumed as measure of the perturbations (see Remark 1.2) and the stability of the trivial solution is then recovered.

1.5 Normal modes analysis

We consider a basic steady solution $\mathbf{v}$ to the I.B.V.P. (1.1)-(1.3) and let $\mathbf{u}$ denote a perturbation to $\mathbf{v}$. The altered motion $\mathbf{v} + \mathbf{u}$ must satisfy the evolution equation (1.1), the same boundary conditions as $\mathbf{v}$, and the initial condition

$$\mathbf{v}(\mathbf{x}, 0) + \mathbf{u}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) + \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega.$$

Thus, for any $T > 0$, the disturbance $\mathbf{u}$ fulfils the *evolution equation of the perturbation*

$$\begin{cases} \mathbf{u}_t = \mathbf{G} & \text{in } \Omega \times (0, T) \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) & \text{in } \Omega \\ A[\mathbf{u} + \mathbf{v}, \nabla(\mathbf{u} + \mathbf{v})] = \mathbf{0} & \text{on } \partial\Omega \times [0, T] \end{cases} \tag{1.18}$$

where

$$\begin{aligned} \mathbf{G} &= \mathbf{G}\left(\mathbf{x}, t, \mathbf{v}, \mathbf{u}, \frac{\partial v_i}{\partial x_r}, \frac{\partial u_i}{\partial x_r}, \frac{\partial^2 v_j}{\partial x_r \partial x_s}, \frac{\partial^2 u_j}{\partial x_r \partial x_s}, \ldots\right) \\ &= \mathbf{F}\left[\mathbf{x}, t, \mathbf{v} + \mathbf{u}, \frac{\partial(v_i + u_i)}{\partial x_r}, \frac{\partial^2(v_j + u_j)}{\partial x_r \partial x_s}, \ldots\right] - \mathbf{F}\left[\mathbf{x}, t, \mathbf{v}, \frac{\partial v_i}{\partial x_r}, \frac{\partial^2 v_j}{\partial x_r \partial x_s}, \ldots\right]. \end{aligned}$$

We now assume that the evolution equations of the perturbations may be linearized for sufficiently small disturbances. The linearization of (1.18) is straightforward in principle and in practise: all products and powers (higher than the first) of the increments are neglected while only the terms which are linear in them are retained. Thereby a linear homogeneous system of partial differential equations is obtained. These have coefficients that may vary in space but not in time because the basic motion is steady. Experience with the method of separation of variables and the Laplace transforms suggests that, in general, the solutions of the linearized system can be expressed as the real parts of integrals of components, each component varying with time like e^{ct} for some complex number $c = c_r + \mathrm{i}c_i$. The linear system will determine the values of c and the spacial variation of corresponding components as eigenvalues and eigenfunctions.

If the basic motion has some simple simmetry, the linear system may be transformed with respect to some of the space variables as well as the time. For example, consider a mechanical system confined between two parallel planes and in which the physical variables in the stationary motion are functions of the coordinate normal to the planes (say z). In this case the Laplace transform of the system with respect to t, the Fourier transforms with respect to x and y may be taken to express the perturbation $\mathbf{u}$ in the form

$$\mathbf{u}(x, y, z, t) = \mathrm{Re} \int_{-\infty}^{+\infty} \mathrm{d}a_x \int_{-\infty}^{+\infty} \mathrm{d}a_y \int_L \hat{\mathbf{u}}(z) \exp[ct + \mathrm{i}(a_x x + a_y y)]\mathrm{d}c$$

where L is the path for the inversion of the Laplace transform. $\hat{\mathbf{u}}$ is to be found from the initial data and the transformed system of ordinary differential equations in z and of the boundary conditions. This system gives an *eigenvalue relation* of the form

$$\mathcal{G}(c, a_x, a_y, X_1, X_2, ..., X_m) = 0$$

which involves the *complex wave speed* c, the *wave numbers* a_x and a_y along the directions x and y, respectively, and other $m(\in \mathbb{N})$ non-dimensional numbers X_i $(i = 1, 2, ..., m)$ related to the basic stationary motion, and further it yields the eigenfunctions $\hat{\mathbf{u}}$ except for an arbitrary function of c, a_x and a_y that may be specified by the initial conditions. This is the method of *normal modes*, whereby small disturbances are expanded in terms of a complete set of modes, which may be treated separately because each satisfies the linear system.

If $c_r > 0$ for a mode, then the corresponding disturbance will be amplified, growing exponentially until it is so large that nonlinearities in the evolution equation of the perturbation become significant. If $c_r = 0$ the mode is said to be *neutrally stable* since the corresponding disturbance will remain small for all time $t > 0$. Finally if $c_r < 0$ the mode is said *strictly stable* or *stable* and the magnitude of corresponding disturbance will tend exponentially to zero as $t \to +\infty$. A small disturbance of the basic motion will in general excite all modes, so that if $c_r > 0$ for at least one mode then the motion is *linearly unstable.* Conversely, if $c_r \leq 0$ for all a complete set of modes then the flow is *linearly stable.* A mode is *marginally stable* if $c_r = 0$ for critical values of the parameters on which the eigenvalue c depends but

$c_r > 0$ for some neighbouring values of the parameters. By definition a marginal stable mode is neutral stable but the converse is not true since, for a neutral stable mode, c_r is not necessarily positive for any neighbouring values of the parameters. The values of the parameters for marginal stability are often sought to give a criterion of stability. The critical relationship among the parameters is the equation of *marginal curve* (or *surface*).

Marginal stable modes can be one of two kinds. The two kinds correspond to the two ways in which the amplitudes of a small perturbation can grow or be damped: they can grow (or be damped) aperiodically; or they can grow (or be damped) by oscillations of increasing (or decreasing) amplitude. In the former case $c = 0$ at marginal stability, i.e. $c_r = c_i = 0$, and the transition from stability to instability takes place via a marginal state exhibiting a stationary pattern of motions. In the latter case $c_i \neq 0$ at marginal stability and the transition takes place via a marginal state exhibiting oscillatory motions with a certain definite characteristic frequency.

If $c_{\mathrm{i}} \neq 0 \Rightarrow c_r < 0$ it then is said that the *Principle of exchange of stabilities* holds. Therefore at the onset of instability a stationary pattern of motion prevails and instability sets in as *steady secondary flow.* Of course if $c_{\mathrm{i}} = 0$ always, i.e. $c \in \mathbb{R}$, then exchange of stabilities always holds such as in the case of the convection cells that arise in a fluid heated from below [11, 17, 87]. On the other hand if $c_{\mathrm{i}} \neq 0$ and exchange of stabilities does not hold, then at the onset of instability oscillatory motions prevail, then one says, according to a definition due to Eddington [18], that one has a case of *overstability.*

1.6 Fundamental topics of nonlinear stability

In this section we shall illustrate some elementary concepts of the theories by Landau and by Hopf by means of the following three examples.

Example 1.6 (Supercritical stability)**.** Let us consider the boundary value problem

$$\begin{cases} u_t - u + u^3 = \dfrac{1}{R} u_{xx} \\ u = 0 \quad \text{at } u = 0 \text{ and } \pi, \end{cases} \tag{1.19}$$

where $R \in \mathbb{R}^+$. We consider the basic steady solution $\bar{u} \equiv 0$. Linearizing the perturbations to the trivial solution, we find

$$\begin{cases} u_t - u = \dfrac{1}{R} u_{xx} \\ u = 0 \quad \text{at } x = 0 \text{ and } \pi, \end{cases}$$

whose solution can easily be represented as sum of the normal modes,

$$u = \sum_{n=1}^{+\infty} A_n \mathrm{e}^{s_n t} \sin(nx),$$

where

$$s_n = 1 - n^2/R \in \mathbb{R} \quad \forall n \in \mathbb{N}$$

and hence the principle of exchange of stability holds. The n-th mode is stable if and only if $R \leq n^2$. Therefore the zero solution is linearly stable if and only if all the modes are stable, i.e.

$$R \leq R_c = \min_{n \in \mathbb{N}} n^2 = 1.$$

Next we examine the nonlinear stability when the parameter R is nearly critical

$$R = R_c + \epsilon = 1 + \epsilon, \quad 0 < |\epsilon| \ll 1.$$

In particular, if R is just supercritical ($\epsilon > 0$), then all the normal modes except the first decay exponentially in time, so it is plausible to ignore the higher modes in the linearized initial-value problem. Accordingly, we approximate the linearized solution by

$$u \cong A_1 \mathrm{e}^{s_1 t} \sin x. \tag{1.20}$$

This solution grows very slowly so that, however small the disturbance is initially, it will cease to be small only after a long time of the order of $-(\ln A_1)/s_1 \sim -(\ln A_1)/\epsilon$ as $R \downarrow 1$. By this time nonlinearity will have modified the exponential growth and the solution given by (1.20) will have become invalid. To approximate the solution uniformly over so long a time, we anticipate that the nonlinear solution satisfies

$$u \sim u_1 \equiv A(t) \sin x \quad \text{as } R \to 1, \ A \to 0 \tag{1.21}$$

for all time, where the amplitude equation is

$$\frac{\mathrm{d}A}{\mathrm{d}t} = a_1 A + a_2 A^2 + a_3 A^3 + \cdots . \tag{1.22}$$

Moreover, we assume that the exact solution can be expanded as

$$u = u_1 + u_2 + u_3 + \cdots , \tag{1.23}$$

where the fundamental mode at marginal stability is given by equation (1.21) and

$$u_r = O(A^r) \quad \text{as } A \to 0 \text{ for } r = 2, 3, \ldots .$$

To find the expansions (1.22) and (1.23) by iteration, we first transfer the small terms of equation $(1.19)_1$ to the right hand side, writing

$$u_{xx} + u = u_t + u^3 + \frac{\epsilon}{1+\epsilon} u_{xx}. \tag{1.24}$$

Note that each of the three terms on the right-hand side is small, the first because the disturbance varies slowly, the second because the nonlinearity is weak, and the third because the parameter R is nearly critical. Note also that the linear operator $\mathrm{L} = \partial^2/\partial x^2 + 1$ associated with the left-hand side in (1.24) is such that $\mathrm{L}u_1 = 0$, where u_1 is the most unstable mode given by equation (1.21) at marginal stability.

Checking the first approximation, we equate all terms of order A in equation (1.24) to find

$$0 = \mathrm{L}u_1 = \left(a_1 - \frac{\epsilon}{1+\epsilon} \right) A \sin x.$$

Therefore we identify $a_1 = s_1 \sim \epsilon$ as $R \to 1$, in agreement with the linear theory.

For the next approximation, we equate terms of order A^2 in equation (1.24) to find in the limit as $R \to 1$ that

$$\mathrm{L}u_2 = a_2 A^2 \sin x. \tag{1.25}$$

Similarly, the boundary conditions $(1.19)_2$ give

$$u_2 = 0 \quad \text{at } x = 0, \pi. \tag{1.26}$$

If the solution u_2 of the linear inhomogeneous problem (1.25)-(1.26) exists, we may multiply (1.25) by u_1, integrate from 0 to π and deduce

$$a_2 A^3 \int_0^\pi \sin^2 x \mathrm{d}x = \int_0^\pi (\mathrm{L}u_2) u_1 \mathrm{d}x = \int_0^\pi u_2 (\mathrm{L}u_1) \mathrm{d}x = 0$$

on integration by parts, and use of the boundary conditions (1.26) and of $\mathrm{L}u_1 = 0$. Therefore

$$a_2 = 0.$$

This is called the *solvability condition* of equations (1.25) and (1.26), it being necessary for the existence of the solution u_2.

We now go back to solve equations (1.25) and (1.26), seeing trivially that

$$u_2 = 0, \tag{1.27}$$

i.e. that the second harmonic happens not to be excited. Of course any multiple of u_1 could be added to this solution u_2, but such an addition could be transferred to the fundamental solution u_1 by re-definition of the amplitude A. So we may take the solution (1.27) without loss of generality. This choice of normalization can be systematized by imposition of the orthogonality condition,

$$\int_0^\pi u_1(u - u_1)\mathrm{d}x = 0. \tag{1.28}$$

For the next approximation, we equate terms of order A^3 in equations (1.24) and $(1.19)_2$, finding in the limit as $R \to 1$ that

$$\mathrm{L}u_3 = (a_3 \sin x + \sin^3 x)A^3 = \left[\left(a_3 + \frac{3}{4}\right)\sin x - \frac{1}{4}\sin(3x)\right]A^3 \tag{1.29}$$

and

$$u_3 = 0 \quad \text{at } z = 0, \pi. \tag{1.30}$$

Multiplying (1.29) by u_1, integrating from 0 to π, etc., we get the solvability condition

$$a_3 = -\frac{3}{4}.$$

Then one may go back to equations (1.29) and (1.30) and show that their solution is

$$u_3 = \frac{A^3}{32}\sin(3x).$$

Although one could go on and find a_4, u_4, a_5, etc. in turn[1] we stop the iteration here, having found the *Landau equation* to the cubic approximation for $0 < |\epsilon| \ll 1$:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \epsilon A - \frac{3}{4}A^3 \tag{1.31}$$

whose explicit general solution is

$$A^2 = \frac{4\epsilon A_0^2}{(4\epsilon - 3A_0^2)\mathrm{e}^{-2\epsilon t} + 3A_0^2}, \tag{1.32}$$

where A_0 is the amplitude at $t = 0$.

First we consider the case $\epsilon < 0$, i.e. R is just subcritical. Then equation (1.32) confirms that the disturbance decays in accord with the linear theory, i.e. $|A| \sim A_0\mathrm{e}^{\epsilon t}$ as $t \to +\infty$ and $A_0 \to 0$. In this case the term $-(3A^3)/4$ in equation (1.31) due to the nonlinearity remains small if it is initially small.

[1] It is easy to show that

$$a_n = 0 \quad \text{and} \quad u_n = 0 \quad \text{for } n = 4, 6, 8, \ldots.$$

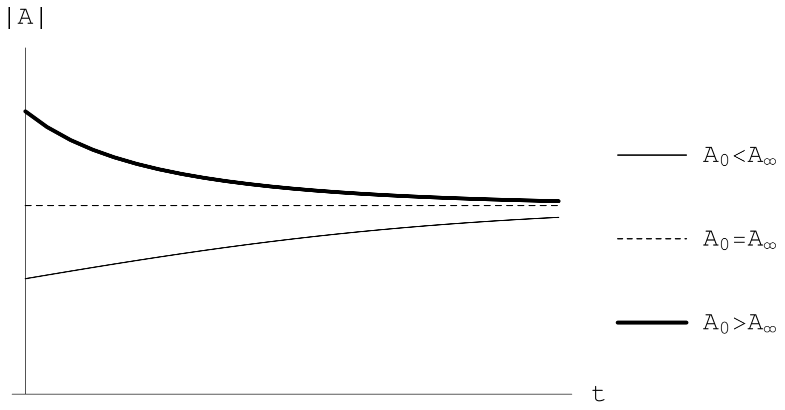

Figure 1.1: Supercritical stability for $0 < \epsilon \ll 1$: the development of $|A|$ as a function of time for two initial values A_0.

If $\epsilon > 0$, i.e. R is just supercritical, it is easy to check that (1.31) admits the stationary solution $|A| = A_\infty = 2\sqrt{\epsilon/3}$ by which we may rewrite (1.32) as

$$A^2 = \frac{A_\infty^2}{\left(\frac{A_\infty^2}{A_0^2} - 1\right) \mathrm{e}^{-2\epsilon t} + 1}$$

and deduce that

$$|A| \to A_\infty \quad \text{as } t \to +\infty,$$

whatever the value of A_0. This is called *supercritical stability*, the basic solution $\bar{u} \equiv 0$ being linearly unstable for $R > 1$ but settling down as a new steady solution which is, moreover, indipendent of the initial conditions. The development of $|A|$ with time is sketched in Figure 1.1 and the dipendence of the steady solutions $|A| = 0$, $|A| = A_\infty$ upon R in Figure 1.2. The branching of the curve of the equilibrium solutions at $R = R_c = 1$ is called a *bifurcation*.

Example 1.7 (Subcritical instability)**.** We now consider the boundary value problem

$$\begin{cases} u_t - u - u^3 = \dfrac{1}{R} u_{xx} \\ u = 0 \quad \text{at } u = 0 \text{ and } \pi, \end{cases}$$

which differs from (1.19) only for the sign of the cubic term. By following the previous arguments, the Landau equation to the cubic approximation for $0 < |\epsilon| \ll 1$ is now given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \epsilon A + \frac{3}{4} A^3, \tag{1.33}$$

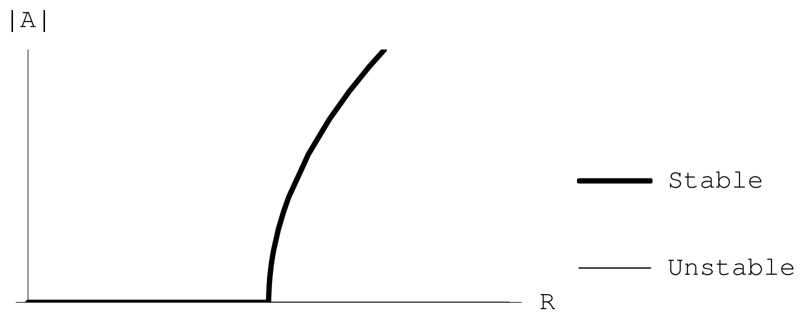

Figure 1.2: The bifurcation curve: the amplitude of the equilibrium solution as a function of R.

whose general solution is

$$A^2 = \frac{4\epsilon A_0^2}{(4\epsilon + 3A_0^2)\mathrm{e}^{-2\epsilon t} - 3A_0^2}, \tag{1.34}$$

where, as before, A_0 is the initial amplitude.

For $\epsilon > 0$ the solution (1.34) breaks down after a finite time, $|A|$ becoming infinite at $t = (2\epsilon)^{-1} \ln[1 + 4\epsilon/(3A_0^2)]$ strengthening the predictions of the linear theory.

For $\epsilon < 0$ (1.33) admits the steady solution $|A| = A_\infty = 2\sqrt{-\epsilon/3}$ and (1.34) may be rewritten as

$$A^2 = \frac{A_\infty^2}{\left(\dfrac{A_\infty^2}{A_0^2} - 1\right)\mathrm{e}^{-2\epsilon t} + 1}$$

by which we readily deduce that

- if $0 < A_0 < A_\infty$, then $|A| \to 0 \quad \text{as } t \to +\infty$;
- if $A_0 = A_\infty$, then $|A| = A_\infty \ \forall t \geq 0$;
- if $A_0 > A_\infty$, then $|A| \to +\infty \quad \text{as } t \to \dfrac{1}{2\epsilon} \ln\left(1 - \dfrac{A_\infty^2}{A_0^2}\right)$.

This case is called *subcritical instability*, because instability occurs with finite amplitude $|A_0| > A_\infty$ when all the infinitesimal disturbances are stable; it is also called *metastability* by physicists. The development of $|A|$ as a function of time is shown in Figure 1.3 and the equilibrium solutions $|A|$ as functions of R in Figure 1.4.

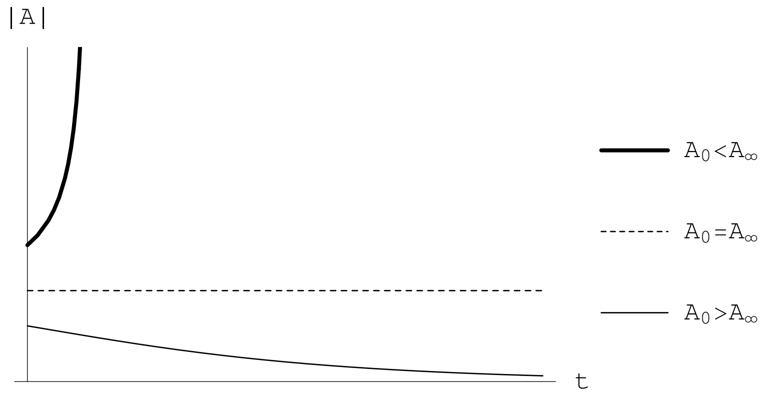

Figure 1.3: Subcritical instability for $0 < -\epsilon \ll 1$: the development of $|A|$ as a function of time for two initial values A_0.

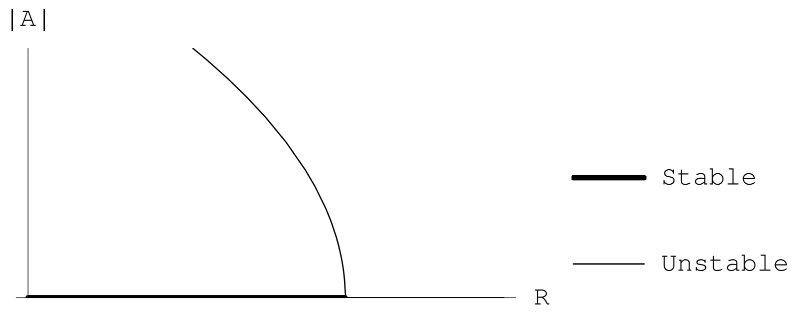

Figure 1.4: The amplitude of the equilibrium solution as a function of R.

Example 1.8 (Hopf bifurcation)**.** We shall now describe the Hopf bifurcation whereby a periodic, rather than a steady, solution may bifurcate at the margin of stability of a steady basic solution. We take as an example the simple system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = (R - x^2 - y^2)x - y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + (R - x^2 - y^2)y, \end{cases} \tag{1.35}$$

where R is a real parameter.

The linear stability of the null solution $x = y = 0$ is governed by the linearized system

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = Rx - y \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = x + Ry. \end{cases} \tag{1.36}$$

Next we find solutions to (1.36) of the type

$$\begin{cases} x = \gamma_1 \mathrm{e}^{ct} \\ y = \gamma_2 \mathrm{e}^{ct} \end{cases} \tag{1.37}$$

with $\gamma_1,\ \gamma_2 \in \mathbb{R}$ and $c = c_r + \mathrm{i}c_i \in \mathbb{C}$. Putting (1.37) into (1.36) yields the following linear algebraic system in the unknowns $\gamma_1,\ \gamma_2$

$$\begin{cases} (c - R)\gamma_1 + \gamma_2 = 0 \\ -\gamma_1 + (c - R)\gamma_2 = 0. \end{cases}$$

For a non-zero solution to this system, we require that

$$(c - R)^2 + 1 = 0$$

by which we get the following eigenvalue relation

$$c = R \pm \mathrm{i}.$$

Then the null solution is linearly stable if $R \leq 0$, unstable if $R > 0$. As R increases through the critical value $R_c = 0$ the real part of the two complex conjugate eigenvalues increases through zero. To examine the nonlinear stability of the trivial solution it seems easiest to use the polar coordinates r and θ, by which $x = r\cos\theta$ and $y = r\sin\theta$, because then the system may be solved explicitly. In fact, equations (1.35) become

$$\begin{cases} \dfrac{\mathrm{d}r}{\mathrm{d}t} = r(R - r^2) \\ \dfrac{\mathrm{d}\theta}{\mathrm{d}t} = 1. \end{cases} \tag{1.38}$$

The first of these equation is a Landau equation (see equation (1.31)), and one may regard x and y as the real and the imaginary part of a complex amplitude $A = r\mathrm{e}^{\mathrm{i}\theta}$. The general solution of system (1.38) is

$$\begin{cases} r^2 = \dfrac{Rr_0^2}{r_0^2 + (R - r_0^2)\mathrm{e}^{-2Rt}} \\ \theta = t + \theta_0 \end{cases} \tag{1.39}$$

where r_0 and θ_0 are the initial values. For $R \leq 0$ all solutions tend to the trivial one as $t \to +\infty$, whatever the initial conditions are, viz the null solution is a global attractor. For $R > 0$, from $(1.39)_1$ we deduce that, if $r_0 = 0$, then $r = 0$ for all $t > 0$, whereas

$$r \to \sqrt{R} \quad \text{as} \quad t \to +\infty \tag{1.40}$$

for $r_0 \neq 0$. Therefore, for $R > 0$, there exists the stable solution

$$\begin{cases} x = \sqrt{R}\cos(t + \theta_0) \\ y = \sqrt{R}\sin(t + \theta_0) \Big(= \sqrt{R}\cos(t + \theta_0 - \pi/2)\Big) \end{cases} \tag{1.41}$$

as well as the unstable null solution. Note that the stable solution (1.41) depends on initial data only through the phase of the complex amplitude A. This is due to the fact that, in this case, the principle of exchange of stabilities does not hold. Finally, from (1.39) and (1.40), for $R > 0$ (x, y) tends to the circle $r = \sqrt{R}$ as $t \to +\infty$ if it initially is any point other than the origin, that is the domain of attraction of the circle $r = \sqrt{R}$ is the whole plane except the origin.

1.7 Liapunov direct method

In 1893 Liapunov introcuced the so-called *direct method* [38] in order to establish conditions ensuring stability of solutions of O.D.E.s and, only in the second half of the fifties, it was generalized to P.D.E.s by Movchan [51, 52]. This approach requires no explicit knowledge of the solutions, but instead it uses an auxiliary function.

Definition 1.9. *Let* $\mathbf{v}$ *a dynamical system on a metric space* X. *A functional* $V : X \to \mathbb{R}$ *is a Liapunov function on a subset* $I \subset X$ *if*

a) V *is continuous on* I,

b) $\forall \mathbf{v}_0 \in I \, : \, V[\mathbf{v}(\mathbf{v}_0, \cdot)]$ *is a non-increasing function of time.*

As observed in Remark 1.1 the stability of a motion on a normed linear space may be expressed through the stability of the zero solution of the perturbed dynamical system. For this reason, one can employ the direct method to investigate the stability of an equilibrium point. Assuming X a normed linear space, denoting by $\mathcal{F}_r$, $r > 0$, the set

$$\mathcal{F}_r = \left\{ f \in \mathrm{C}^0([0,r)) : f(0) = 0,\ f \text{strictly increasing} \right\},$$

and by E_α, $\alpha \in \mathbb{R}$, the set

$$E_\alpha = \{\mathbf{x} \in X : V(\mathbf{x}) < \alpha\},$$

then the Liapunov direct method can be summarized by the following two Theorems.

Theorem 1.2. *Let* $\mathbf{u}$ *be a dynamical system on a normed linear space* X *and let* $\mathbf{0}$ *be an equilibrium point. If* V *is a Liapunov function on the open ball* $S(\mathbf{0}, r)$*, for some* $r > 0$*, such that:*

i) $V(\mathbf{0}) = 0$*,*

ii) $\exists f \in \mathcal{F}_r \ :\ V(x) \geq f(\|x\|)\ \forall \mathbf{x} \in S(\mathbf{0}, r)$*,*

then $\mathbf{0}$ *is stable. If, in addition,*

iii) $\forall \mathbf{x} \in S(\mathbf{0}, r)\ V[\mathbf{u}(\mathbf{x}, \cdot)]$ *is differentiable with respect to time,*

iv) $\exists g \in \mathcal{F}_r : \dot{V}[\mathbf{u}(\mathbf{x}, t)] \leq -g(\|\mathbf{u}(\mathbf{x}, t)\|)\ \forall \mathbf{x} \in S(\mathbf{0}, r),\ \forall t \in \mathbb{R}^+$*,*

then $\mathbf{0}$ *is asymptotically stable.*

Proof. Let us consider $0 < \epsilon < r$ and introduce $\alpha \in \mathbb{R}$,

$$0 < \alpha < f(\epsilon) \leq \inf_{\|\mathbf{x}\|=\epsilon} V(\mathbf{x}).$$

By *ii*) and the continuity of V on $S(\mathbf{0}, r)$ we readily deduce that E_α is an open positive invariant subset of the open ball $S(\mathbf{0}, \epsilon)$. The stability is then immediately obtained observing that, by the assumption *i*) and the continuity of V in $\mathbf{0}$, there exists $\delta(\epsilon) > 0$ such that $S(\mathbf{0}, \delta) \subset E_\alpha$ and so $\mathbf{u}_0 \in S(\mathbf{0}, \delta) \Rightarrow \gamma(\mathbf{u}_0) \subset E_\alpha \subset S(\mathbf{0}, \epsilon)$.

Concerning the asymptotic stability, choosing $\mathbf{u}_0 \in S(\mathbf{0}, \delta)$, by *ii*)-*iv*) it follows that

$$\begin{aligned} 0 \leq f(\|\mathbf{u}(\mathbf{u}_0, t)\|) \leq V[\mathbf{u}(\mathbf{u}_0, t)] &= V(\mathbf{u}_0) + \int_0^t \dot{V}[\mathbf{u}(\mathbf{u}_0, \tau)]\mathrm{d}\tau \\ &\leq V(\mathbf{u}_0) - \int_0^t g(\|\mathbf{u}(\mathbf{u}_0, \tau)\|)\mathrm{d}\tau \quad \forall t \in \mathbb{R}^+. \end{aligned} \tag{1.42}$$

Since $V[\mathbf{u}(\mathbf{u}_0, \cdot)] : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded non-increasing function, there exists $\beta \in \mathbb{R}$ such that

$$0 \leq \inf_{t \geq 0} V[\mathbf{u}(\mathbf{u}_0, t)] = \beta \leq V(\mathbf{u}_0) < \alpha.$$

But $\beta > 0$ implies $\gamma(\mathbf{u}_0) \cap E_\beta = \emptyset$ and hence, since E_β is an open set, there exists $r^* > 0$ such that $S(\mathbf{0}, r^*) \subset E_\beta$ and

$$\|\mathbf{u}(\mathbf{u}_0, t)\| \geq r^*, \quad g(r^*) \leq g(\|\mathbf{u}(\mathbf{u}_0, t)\|) \ \forall t \geq 0.$$

Consequently (1.42) gives

$$0 < V[\mathbf{u}(\mathbf{u}_0, t)] \leq V(\mathbf{u}_0) - \int_0^t g(r^*) \mathrm{d}\tau = V(\mathbf{u}_0) - t g(r^*) < 0$$

for $t > V(\mathbf{u}_0)/g(r^*)$, which is impossible. Therefore $\beta = 0$ and the asymptotic stability is then achieved. □

Theorem 1.3. *Let* $\mathbf{u}$ *be a dynamical system on the normed linear space* X *and* $\mathbf{0}$ *be an equilibrium point. If* V *is a Liapunov function on the open ball* $S(\mathbf{0}, r)$*, for some* $r > 0$*, and*

i) $V(\mathbf{0}) = 0$,

ii) $\forall \epsilon \in]0, r]$ *the open set* $A_\epsilon = S(\mathbf{0}, \epsilon) \cap E_0$ *is non-empty,*

iii) $\forall \mathbf{x} \in A_r$ $V[\mathbf{u}(\mathbf{x}, \cdot)]$ *is differentiable with respect to time,*

iv) $\exists g \in \mathcal{F}_{\bar{r}}$, $\bar{r} > -\inf_{A_r} V$*, such that*

$$\dot{V}[\mathbf{u}(\mathbf{x}, t)] \leq -g[-V[\mathbf{u}(\mathbf{x}, t)]] \quad \forall \mathbf{x} \in A_r, \ \forall t \in \mathbb{R}^+,$$

then $\mathbf{0}$ *is unstable.*

Proof. Because of *i*) and the continuity of V in $\mathbf{0}$, there exists $\epsilon \in]0, r]$ such that $\mathbf{x} \in S(\mathbf{0}, \epsilon) \Rightarrow V(\mathbf{x}) > -1$. The equilibrium point $\mathbf{0}$ cannot be stable because, otherwise, there would exist $\delta \in]0, \epsilon]$ such that $\mathbf{u}_0 \in S(\mathbf{0}, \delta) \Rightarrow \gamma(\mathbf{u}_0) \subset S(\mathbf{0}, \epsilon)$ and hence, taking into account *ii*), $\mathbf{u}_0 \in A_\delta \Rightarrow \gamma(\mathbf{u}_0) \subset A_\epsilon$. Next, chosen $\mathbf{u}_0 \in A_\delta$, *iii*) and *iv*) give

$$\begin{aligned} -1 < V[\mathbf{u}(\mathbf{u}_0, t)] &= V(\mathbf{u}_0) + \int_0^t \dot{V}[\mathbf{u}(\mathbf{u}_0, \tau)] \mathrm{d}\tau \\ &\leq V(\mathbf{u}_0) - \int_0^t g[-V(\mathbf{u}(\mathbf{u}_0, \tau))] \mathrm{d}\tau \\ &\leq V(\mathbf{u}_0) - \int_0^t g[-V(\mathbf{u}_0)] \mathrm{d}\tau = V(\mathbf{u}_0) - t g[-V(\mathbf{u}_0)] < -1 \end{aligned}$$

for $t > \dfrac{1 + V(\mathbf{u}_0)}{g[-V(\mathbf{u}_0)]}$, which is impossible. Therefore $\mathbf{0}$ is unstable. □

Remark 1.3. Let $\mathbf{u}$ be a dynamical system on a normed linear space X and $\mathbf{0}$ be an equilibrium point. If V is a Liapunov function on $S(\mathbf{0}, r)$ and satisfies

$$V(\mathbf{0}) = 0, \quad V(\mathbf{x}) > 0 \ \forall \mathbf{x} \neq \mathbf{0},$$

then $\mathbf{0}$ is stable with respect to the measure V of the perturbation (see Remark 1.2). Moreover, if there exists $c > 0$ such that

$$\dot{V} \leq -cV$$

along the motions with initial data in $S(\mathbf{0}, r)$, then $\mathbf{0}$ is asymptotically exponentially stable with respect to the measure V of pertubation according to the following inequality

$$V[\mathbf{u}(\mathbf{u}_0, t)] \leq V(\mathbf{u}_0)\mathrm{e}^{-ct} \quad \forall \mathbf{u}_0 \in S(\mathbf{0}, r), \ \forall t \in \mathbb{R}^+.$$

1.8 The norm as Liapunov function: the energy method

Although *energy method* originated in the works of Reynolds [71] and Orr [59], its modern version can be considered to be a particular case of the Liapunov direct method (see [23, 81, 87]).

Let us consider a basic solution $\mathbf{v}$ to the I.B.V.P. (1.1)-(1.3) and deduce the evolution equations of the perturbation (1.18) as in section 1.5. Then we take a linear subspace $\mathcal{H}$ of $\mathrm{L}^2(\Omega)$, endowed with the standard L^2-norm

$$\|f\|^2 = \int_\Omega f^2 \mathrm{d}\Omega,$$

as state space and define the *energy* E of the perturbation $\mathbf{u}$ through

$$E(t) = \frac{1}{2}\|\mathbf{u}\|^2.$$

For the fluid motions treated in this thesis multiplying $(1.18)_1$ by $\mathbf{u}$ and integrating over Ω yield a relation of the type

$$\dot{E}(t) = R\mathcal{I} - \mathcal{D}$$

where R is a non-dimensional number related to the physics of the phenomenon $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{D}$ are quadratic integral functionals involving $\mathbf{u}$ and $\nabla\mathbf{u}$, $\mathcal{D}$ being definite positive. The stability of the basic solution $\mathbf{v}$ is then linked to the variational problem

$$\frac{1}{R_E} = \max_{\mathcal{H}} \frac{\mathcal{I}}{\mathcal{D}}.$$

Indeed, if a Poincaré type inequality holds, that is

$$\exists \gamma \in \mathbb{R}^+ \;:\; \|\mathbf{u}\| \leq \gamma \mathcal{D} \quad \forall \mathbf{u} \in \mathcal{H},$$

one obtains the *energy inequality*

$$\dot{E}(t) \leq -\frac{2}{\gamma}\frac{R_E - R}{R_E} E(t)$$

which integrates to

$$E(t) \leq E(0) \exp\left(-\frac{2}{\gamma}\frac{R_E - R}{R_E} t\right)$$

and consequently the global exponential stability of the basic motion $\mathbf{v}$ is achieved (see Remark 1.3).

Example 1.9. We end this introductory chapter by illustrating the energy method on a simple example.

We wish to examine the stability of the zero solution to the I.B.V.P. for the diffusion equation with a linear source term and a convective term

$$\begin{cases} u_t + uu_x = u_{xx} + au & x \in (0, d), \;\; t \in \mathbb{R}^+ \\ u(0, t) = u(d, t) = 0 & \forall t \geq 0 \\ u(x, 0) = u_0(x) & \forall x \in (0, d) \end{cases} \tag{1.43}$$

where a is a non-negative constant.

If we attempt a linear analysis, that is linearize $(1.43)_1$ about the trivial solution $u \equiv 0$, we obtain

$$\begin{cases} u_t = u_{xx} + au & x \in (0, d), \;\; t \in \mathbb{R}^+ \\ u(0, t) = u(d, t) = 0 & \forall t \geq 0 \\ u(x, 0) = u_0(x) & \forall x \in (0, d) \end{cases}$$

whose solution may be represented as an infinite sum of normal modes

$$u = \sum_{n=1}^{+\infty} A_n \mathrm{e}^{s_n t} \sin\frac{n\pi x}{d}$$

where

$$s_n = a - \frac{n^2\pi^2}{d^2} \quad \text{and} \quad A_n = \frac{2}{d}\int_0^d u_0(x) \sin\frac{n\pi x}{d} \mathrm{d}x.$$

The n-th mode is stable if and only if $a \le n^2\pi^2/d^2$ and so the zero solution is linearly stable if and only if all modes are stable, namely if and only if

$$a \le \min_{n\in\mathbb{N}} \frac{n^2\pi^2}{d^2} = \frac{\pi^2}{d^2}. \tag{1.44}$$

In order to find a sufficient condition for global nonlinear stability of the zero solution we employ the energy method. Then we multiply the differential equation $(1.43)_1$ by u, integrate over the interval $(0, d)$ and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = \int_0^d u\frac{\partial^2 u}{\partial x^2}\mathrm{d}x + a\|u\|^2$$

where $\|\cdot\|$ denotes the standard $\mathrm{L}^2(0, d)$-norm. Note that the convective term integrates to zero, since, from $(1.43)_2$,

$$\int_0^d u^2\frac{\partial u}{\partial x}\mathrm{d}x = \frac{1}{3}u^3\big|_0^d = 0.$$

Again, as before, integrating by parts yields

$$\int_0^d u\frac{\partial^2 u}{\partial x^2}\mathrm{d}x = -\|u_x\|^2.$$

So we get the energy inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = -\|u_x\|^2 + a\|u\|^2 \le -\|u_x\|^2\left(1 - a\max_{\mathcal{H}}\frac{\|u\|^2}{\|u_x\|^2}\right), \tag{1.45}$$

where $\mathcal{H} = \{u \in \mathrm{H}^1(0, d) : u(0) = u(d) = 0\}$ is *the space of the admissible functions* over which we seek a maximum. We now define R_E by

$$\frac{1}{R_E} = \max_{\mathcal{H}}\frac{\|u\|^2}{\|u_x\|^2}, \tag{1.46}$$

so that the energy inequality (1.45) may be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = -2\|u_x\|^2\left(1 - \frac{a}{R_E}\right).$$

If $a < R_E$, then $1 - a/R_E > 0$ and, by using the Poincaré inequality, we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = -2\frac{\pi^2}{d^2}\left(1 - \frac{a}{R_E}\right)\|u\|^2$$

and consequently

$$\|u\|^2 \le \|u_0\|^2 \exp\left[-2\frac{\pi^2}{d^2}\left(1 - \frac{a}{R_E}\right)t\right].$$

We have thus shown that if $a < R_E$, $\|u\| \to 0$ as $t \to +\infty$ with the decay at least exponential in time.

The problem remains to find R_E. The Euler-Lagrange equation associated to the variational problem (1.46) is found as follows. Let $u \in \mathcal{H}$ be the function on which $\|u\|^2/\|u_x\|^2$ attains its maximum. Then, letting ϵ be a non-negative parameter, for any $\phi \in \mathcal{H}$, the function

$$F(\epsilon) = \frac{\|u + \epsilon\phi\|^2}{\|u_x + \epsilon\phi_x\|^2}$$

attains its maximum for $\epsilon = 0$ and hence one has

$$\begin{aligned} 0 = \left.\frac{\mathrm{d}F}{\mathrm{d}\epsilon}\right|_{\epsilon=0} &= \frac{2}{\|u_x\|^2}\left(\int_0^d u\phi \mathrm{d}x - \frac{\|u\|^2}{\|u_x\|^2}\int_0^d u_x\phi_x \mathrm{d}x\right) \\ &= \frac{2}{\|u_x\|^2}\int_0^d \left(u\phi - R_E^{-1}u_x\phi_x\right)\mathrm{d}x = \int_0^d \phi\left(u + R_E^{-1}u_{xx}\right)\mathrm{d}x. \end{aligned}$$

Since ϕ is an arbitrary function belonging to $\mathcal{H}$, we have

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + R_E u = 0, \quad u(0) = u(d) = 0 \tag{1.47}$$

which gives an eigenvalue problem for R_E.

It is easy to show that the eigenvalues of (1.47) are given by

$$R_E = \frac{n^2\pi^2}{d^2}, \quad n \in \mathbb{N}.$$

For stability, we need $a < \min_{n\in\mathbb{N}} R_E = \pi^2/d^2$. Therefore the criterion we have just derived by employing the energy method is the same as that found by normal modes analysis and so (1.44) is a necessary and sufficient condition for stability of the zero solution to the I.B.V.P. (1.43). Furthermore, since the condition $a < \pi^2/d^2$ ensures global nonlinear exponential stability of the zero solution, no subcritical instability is allowable (see Example 1.7).

Chapter 2

Equations of Fluid Mechanics

2.1 Equations of balance of continuum mechanics

Let $Ox_1x_2x_3$ be a Cartesian frame of reference with fundamental unit vectors $\mathbf{e}_i$ $(i = 1, 2, 3)$, $\mathbf{e}_3$ pointed vertically upward, and let V be a volume whose surface ∂V moves with velocity $\mathbf{v} = v_j \mathbf{e}_j$ of a body. Therefore at time t the rate of change of a generic quantity

$$\Psi = \int_V \rho\psi \mathrm{d}V$$

inside V is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\psi \mathrm{d}V = \int_V \frac{\partial(\rho\psi)}{\partial t}\mathrm{d}V + \int_{\partial V} \rho\psi v_j n_j \mathrm{d}A \tag{2.1}$$

where $\rho\psi$ is the density of the quantity Ψ, ψ being the specific value of Ψ, and $\mathbf{n} = n_j \mathbf{e}_j$ is the outer unit normal. Equation (2.1) is known as *Reynolds' transport theorem* [25].

The quantity Ψ may change in time due to a flux of Ψ through the surface ∂V, due to a production of Ψ and due to a supply from outside. For V the rate of change of Ψ may be expressed by the generic equation of balance

$$\int_V \frac{\partial(\rho\psi)}{\partial t}\mathrm{d}V = -\int_{\partial V} (\rho\psi v_j + \Phi_j) n_j \mathrm{d}A + \int_V \pi \mathrm{d}V + \int_V \rho\sigma \mathrm{d}V, \tag{2.2}$$

where $\mathbf{\Phi}$ is the non-convective flux density vector of Ψ, π is the production density and σ is the specific supply from outside. Given the appropriate smoothness properties, the surface integral in (2.2) may be converted into a volume integral by use of the Gauss Theorem and then (2.2) may be written as

$$\int_V \left[\frac{\partial(\rho\psi)}{\partial t} + \frac{\partial(\rho\psi v_j + \Phi_j)}{\partial x_j} - \pi - \rho\sigma\right] \mathrm{d}V = 0.$$

Since this equation must hold for all volumes, even infinitesimally small ones, the integrand itself must vanish. Thus we obtain the generic local equation of balance

$$\frac{\partial(\rho\psi)}{\partial t} + \frac{\partial(\rho\psi v_j + \Phi_j)}{\partial x_j} - \pi - \rho\sigma = 0. \tag{2.3}$$

The prototype of equation (2.3) is the mass balance which results from setting $\psi = 1$ and $\Phi_j = \pi = \sigma = 0$ so that we obtain

$$\frac{\partial \rho}{\partial t} + \frac{(\rho v_j)}{\partial x_j} = 0,$$

which is known as the continuity equation and may be used to simplify the generic local balance equation (2.3) to read

$$\rho\dot{\psi} + \operatorname{div}\boldsymbol{\Phi} = \pi + \rho\sigma,$$

where

$$\dot{\psi} = \frac{\partial \psi}{\partial t} + v_j \frac{\partial \psi}{\partial x_j} \tag{2.4}$$

is the *material derivative* of ψ.

The most commonly appearing balance equations of continuum mechanics are those of mass, linear momentum and internal energy. In those cases the generic quantities ψ, Φ_j, π and σ have concrete physical significance and are all denoted by canonical letters. Table 2.1 gives a list.

Ψ	ψ	Φ_i	π	σ
mass	1	0	0	0
linear momentum	v_i	$-t_{ij}$	0	b_i
internal energy	e	q_j	$t_{ij}d_{ij}$	r

Table 2.1: Canonical notation for specific values of mass, linear momentum and internal energy and their fluxes and source contributions.

$\mathbf{T} = t_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ in the flux density of linear momentum is the *Cauchy stress tensor* and the flux density $\mathbf{q}$ of internal energy is called the *heat flux vector*. In absence of body couples, the balance of the angular momentum requires that the stress tensor $\mathbf{T}$ is symmetric, i.e. $t_{ij} = t_{ji}\,\forall i = 1, 2, 3$. The external supply $\mathbf{b}$ of linear momentum is the *specific external body force field* and the supply r of internal energy is the *specific radiant heating*. Finally the second order tensor

$$\mathbf{D} = d_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

in the production density of internal energy is the symmetric part of velocity gradient $\mathbf{L} = \nabla \mathbf{v}$. Then, according to (2.3) and (2.4), the equations of balance of mass, linear momentum and internal energy are:

$$\dot{\rho} + \rho \mathrm{div}\, \mathbf{v} = 0, \tag{2.5}$$

$$\rho \dot{\mathbf{v}} = \mathrm{div}\, \mathbf{T} + \rho \mathbf{b}, \tag{2.6}$$

$$\rho \dot{e} + \mathrm{div}\, \mathbf{q} = \mathbf{T} \cdot \mathbf{D} + \rho r. \tag{2.7}$$

2.2 Constitutive assumptions for fluid behavior

The equations of balance (2.5)-(2.7) are common to most bodies in Nature. These laws, however, are insufficient to fully characterize the behavior of bodies because they do not distinguish between different types of materials. We therefore introduce additional hypothesis, called *constitutive assumptions*, which serve to distinguish different types of material behavior.

Here we shall consider three types of constitutive assumptions in order to describe the fluid behavior.

(i) Constraint on the possible deformations the fluid may undergo.

(ii) Assumptions on the form of the stress tensor.

(iii) Constitutive equations relating the material parameters of the fluid to the motion.

From now on we shall be interested in isotropic linearly viscous fluids that can only undergo isochoric motions in isothermal processes, but can sustain motions that are not necessarily isochoric in processes that are not isothermal. Such fluids are said to be, roughly speaking, mechanically incompressible but thermally compressible. Experience tells us the possibility that a fluid be mechanically incompressible but thermally compressible seems a reasonable description of observations. The restriction that the fluid can undergo only isochoric motions in isothermal processes implies that the determinant of the deformation gradient is a function of temperature θ,

$$\det \mathbf{F} = f(\theta). \tag{2.8}$$

If $\mathbf{F}$ is differentiable with respect to time, (2.8) can be expressed as

$$\mathrm{div}\, \mathbf{v} = \mathrm{tr}\mathbf{D} = \alpha(\theta)\dot{\theta} \tag{2.9}$$

where

$$\alpha(\theta) = \frac{1}{f(\theta)} \frac{\mathrm{d}f}{\mathrm{d}\theta}(\theta)$$

is the *coefficient of volumetric thermal expansion.*

Constitutive expressions for the stress within the context of classical continuum mechanics such as those for the linearized response of solids due to Hooke and Navier, and for the linear response of fluids due to Newton, Navier, Poisson, St. Venant and Stokes provide explicit relationships for the stress in terms of appropriate kinematical quantities and the density. For instance, in the case of the classical incompressible Navier-Stokes fluid the Cauchy stress tensor takes the explicit form

$$\mathbf{T} = -p\mathbf{1} + 2\mu(\theta)\mathbf{D}, \tag{2.10}$$

where $-p\mathbf{1}$ is the indeterminate part of the stress due to the constraint of incompressibility (i.e. the constraint stress), p being the *pressure*, and μ is the *viscosity* of the fluid. In contrast, many constitutive relations for inelastic and viscoelastic fluids are implicit relations. Here, following Rajagopal [65], we shall discuss a generalization of the classical incompressible Navier-Stokes fluid, as envisioned by Stokes [85], that leads to implicit constitutive relations.

In his celebrated paper on the response of fluids Stokes [85] recognized that the viscosity of a fluid could depend upon the pressure. However, based on the experiments of Du Buat on the flow of water in canals and pipes under normal operating conditions, Stokes suggested that the viscosity could be considered a constant for such flows. Stokes was however very careful to delineate the class of flows wherein viscosity might be considered a constant and he also remarked that such an assumption would be invalid under other flow conditions. As early as Barus [5] proposed an empirical relationship between the viscosity and the pressure, namely

$$\mu(p) = \mu_0 \exp[\beta(p - p_0)], \tag{2.11}$$

where μ_0 is the viscosity at the pressure p_0 and β is a piezoviscous coefficient that varies with temperature. Later, Andrade [2] suggested the following expression for the viscosity

$$\mu(p, \rho, \theta) = A\rho^{1/2} \exp\left[(p + \rho r^2)\frac{s}{T}\right],$$

based on experiments. In the above expression ρ denotes the density, T the temperature, p the pressure, and r, s and A are constants. More recently, Laun [36] has modelled the viscosity of polymer melts through

$$\mu(p, T) = \mu_0 \exp[\beta(p - p_0) - \gamma(T - T_0)],$$

where μ_0 is the viscosity at pressure p_0 and temperature T_0, and β and γ are non-negative constants. There have been numerous other experiments by Bair and co-workers that shows that the dependence of the viscosity on the pressure is exponential (see recent experiments of Bair and Kottke [4]). Mention must be made of the works of Martín-Alfonso and co-workers [46, 47] wherein intricate relationship among the temperature, viscosity and pressure are provided for bitumen.

In order to deduce the model (2.10), the standard procedure in classical mechanics is to split the Cauchy stress tensor $\mathbf{T}$ additively as

$$\mathbf{T} = \mathbf{T}_C + \mathbf{T}_E, \tag{2.12}$$

where $\mathbf{T}_C$, the constraint stress, is assumed not to depend on the state variables (in the case of the classical fluid the velocity gradient) and $\mathbf{T}_E$, the so-called 'extra' stress, is constitutively prescribed, but is assumed to not depend on the constrained part $\mathbf{T}_C$. According to the the Constraint Principle of Truesdell and Noll [90], the further assumption that $\mathbf{T}_C$ does no work implies that

$$\mathbf{T}_C \cdot \mathbf{D} = 0 \quad \text{whenever } \operatorname{tr}\mathbf{D} = \mathbf{1} \cdot \mathbf{D} = 0.$$

This immediately leads to

$$\mathbf{T}_C = -p\mathbf{1},$$

p being a Lagrange multiplier. Importantly, $\mathbf{T}_E$ cannot depend on p, and thus quantities such as the viscosity cannot depend on the pressure. It is also important to note that the above procedure would be inapplicable if the constraint were nonlinear in $\mathbf{D}$. In any event, the standard procedure leads to the material function not depending on the constraint.

Let us consider an implicit relation of the form

$$\mathbf{f}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta}) = \mathbf{0}, \tag{2.13}$$

i.e. among the stress, the symmetric part of the velocity gradient, the temperature and the material derivative of the temperature. It then follows that

$$\frac{\partial \mathbf{f}}{\partial \mathbf{T}}\dot{\mathbf{T}} + \frac{\partial \mathbf{f}}{\partial \mathbf{D}}\dot{\mathbf{D}} + \frac{\partial \mathbf{f}}{\partial \theta}\dot{\theta} + \frac{\partial \mathbf{f}}{\partial \dot{\theta}}\ddot{\theta} = \mathbf{0},$$

where $\partial\mathbf{f}/\partial\mathbf{T}$ and $\partial\mathbf{f}/\partial\mathbf{D}$ are fourth-order tensors, $\partial\mathbf{f}/\partial\theta$ and $\partial\mathbf{f}/\partial\dot{\theta}$ are second-order tensors. We could also start with models of the form

$$\begin{aligned}[\mathbf{A}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta})]\dot{\mathbf{T}} + [\mathbf{B}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta})]\dot{\mathbf{D}} + \mathbf{C}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta})\dot{\theta} \\ + \mathbf{E}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta})\ddot{\theta} = \mathbf{0},\end{aligned} \tag{2.14}$$

where $\mathbf{A}$ and $\mathbf{B}$ are fourth-order tensor, $\mathbf{C}$ and $\mathbf{E}$ are second-order tensor. While the class of models defined through (2.14) is larger, in one sense, than that defined through (2.13) since not all models belonging to (2.14) belong to (2.13) as (2.14) may not be integrable, we note that (2.14) requires the stress $\mathbf{T}$, the symmetric velocity gradient $\mathbf{D}$ and the material time derivative of temperature $\dot{\theta}$ have time derivatives while (2.13) makes no such restriction. However, we shall be interested in sufficiently smooth functions $\mathbf{T}$, $\mathbf{D}$ and θ, so for such a class of functions (2.14) is more general than (2.13). Given an explicit model for the Cauchy stress tensor, since it can always be expressed in the form (2.13), we can express it in the form (2.14) by merely taking its derivative.

Suppose

$$\begin{cases} \mathbf{A}(\mathbf{T},\mathbf{D},\theta,\dot{\theta}) = \boldsymbol{\mathcal{I}} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1} - 2\dfrac{\partial\mu}{\partial\mathrm{tr}\mathbf{T}}(\mathrm{tr}\mathbf{T},\theta)\left[\mathbf{D} - \dfrac{\alpha(\theta)\dot{\theta}}{3}\mathbf{1}\right]\otimes\mathbf{1}, \\ \mathbf{B}(\mathbf{T},\mathbf{D},\theta,\dot{\theta}) = -2\mu(\mathrm{tr}\mathbf{T},\theta)\boldsymbol{\mathcal{I}}, \\ \mathbf{C}(\mathbf{T},\mathbf{D},\theta,\dot{\theta}) = -2\dfrac{\partial\mu}{\partial\theta}(\mathrm{tr}\mathbf{T},\theta)\left[\mathbf{D} - \dfrac{\alpha(\theta)\dot{\theta}}{3}\mathbf{1}\right] + \frac{2}{3}\mu(\mathrm{tr}\mathbf{T},\theta)\frac{\mathrm{d}\alpha}{\mathrm{d}\theta}(\theta)\dot{\theta}\mathbf{1}, \\ \mathbf{E}(\mathbf{T},\mathbf{D},\theta,\dot{\theta}) = \frac{2}{3}\mu(\mathrm{tr}\mathbf{T},\theta)\alpha(\theta)\mathbf{1}, \end{cases} \tag{2.15}$$

where $\boldsymbol{\mathcal{I}}$ denotes the fourth-order identity tensor, μ and α are sufficiently smooth functions, μ depending on both $\mathrm{tr}\mathbf{T}$ and θ, α only on θ. Furthermore, since we are interested in describing mechanically incompressible but thermally compressible fluids, we shall require that (2.9) is met.

From (2.14) and (2.15) it follows that

$$\begin{aligned} \dot{\mathbf{T}} =& \frac{1}{3}(\mathrm{tr}\dot{\mathbf{T}})\mathbf{1} + 2\left(\frac{\partial\mu}{\partial\mathrm{tr}\mathbf{T}}\mathrm{tr}\dot{\mathbf{T}} + \frac{\partial\mu}{\partial\theta}\dot{\theta}\right)\mathbf{D} + 2\mu(\mathrm{tr}\mathbf{T},\theta)\dot{\mathbf{D}} \\ &- \frac{2}{3}\mu(\mathrm{tr}\mathbf{T},\theta)\left[\frac{\mathrm{d}\alpha}{\mathrm{d}\theta}(\theta)\dot{\theta}^2 + \alpha(\theta)\ddot{\theta}\right]\mathbf{1} - \frac{2}{3}\alpha(\theta)\dot{\theta}\left(\frac{\partial\mu}{\partial\mathrm{tr}\mathbf{T}}\mathrm{tr}\dot{\mathbf{T}} + \frac{\partial\mu}{\partial\theta}\dot{\theta}\right)\mathbf{1}, \end{aligned}$$

which can be integrated to yield

$$\mathbf{T} = \frac{1}{3}(\mathrm{tr}\mathbf{T})\mathbf{1} + 2\mu(\mathrm{tr}\mathbf{T},\theta)\left[\mathbf{D} - \frac{1}{3}\alpha(\theta)\dot{\theta}\mathbf{1}\right] + \mathbf{T}_0$$

where $\mathbf{T}_0$ is some constant symmetric stress tensor. The further requirement that the stress be purely spherical when the fluid is at rest in isothermal processes leads to

$$\mathbf{T} = \frac{1}{3}(\mathrm{tr}\mathbf{T})\mathbf{1} + 2\mu(\mathrm{tr}\mathbf{T},\theta)\left[\mathbf{D} - \frac{1}{3}\alpha(\theta)\dot{\theta}\mathbf{1}\right]. \tag{2.16}$$

We notice that (2.16) automatically meets the constraint (2.9). We thus do not need to enforce the constraint (2.9) by using a Lagrange multiplier.

Let us define

$$p = -\frac{1}{3}\mathrm{tr}\mathbf{T},$$

then, by (2.9) and (2.16),

$$\mathbf{T} = -p\mathbf{1} + 2\mu(p,\theta)\left[\mathbf{D} - \frac{1}{3}(\mathrm{tr}\mathbf{D})\mathbf{1}\right]. \tag{2.17}$$

We now consider the implications of assuming that $\mathbf{f}$ defined through the relation (2.13) is an isotropic function. Then

$$\mathbf{f}(\mathbf{QTQ}^T, \mathbf{QDQ}^T, \theta, \dot{\theta}) = \mathbf{Qf}(\mathbf{T}, \mathbf{D}, \theta, \dot{\theta})\mathbf{Q}^T \quad \forall \mathbf{Q} \in \mathrm{Orth},$$

where Orth denotes the set of all orthogonal transformations. It then follows that (see Spencer [82])

$$\begin{aligned} &\alpha_0\mathbf{1} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{TD} + \mathbf{DT}) \\ &\quad + \alpha_6(\mathbf{T}^2\mathbf{D} + \mathbf{DT}^2) + \alpha_7(\mathbf{TD}^2 + \mathbf{D}^2\mathbf{T}) + \alpha_8(\mathbf{T}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}^2) = \mathbf{0}, \end{aligned} \tag{2.18}$$

where the material functions α_i, $i = 0, 1, \ldots, 8$, depend on θ, $\dot{\theta}$ and on the invariants

$$\mathrm{tr}\,\mathbf{T}, \mathrm{tr}\,\mathbf{D}, \mathrm{tr}\,\mathbf{T}^2, \mathrm{tr}\,\mathbf{D}^2, \mathrm{tr}\,\mathbf{T}^3, \mathrm{tr}\,\mathbf{D}^3, \mathrm{tr}\,(\mathbf{TD}), \mathrm{tr}\,(\mathbf{T}^2\mathbf{D}), \mathrm{tr}\,(\mathbf{TD}^2), \mathrm{tr}\,(\mathbf{T}^2\mathbf{D}^2).$$

When we consider fluid models of the form (2.18), if

$$\alpha_0 = -\frac{1}{3}\mathrm{tr}\mathbf{T} + \frac{2}{3}\mu(\mathrm{tr}\mathbf{T}, \theta)\alpha(\theta)\dot{\theta}, \quad \alpha_1 = 1, \quad \alpha_2 = -2\mu(\mathrm{tr}\mathbf{T}, \theta)$$

and, as we are interested in linearly viscous fluids, all the other α_i are identically zero, we obtain the model (2.17). Such a constitutive assumption, i.e. the special choice of the functions α_i ($i = 0, 1, \ldots, 8$), automatically implies that the fluid under consideration is mechanically incompressible but thermally compressible as it always meets the constraint (2.9). We may then conclude that we do not need to necessarily enforce the constraint via Lagrange multipliers or require that the constraint stress is workless while working with these implicit models.

We shall henceforth take (2.17) as model for the Cauchy stress tensor.

For a fluid it is customary to require constitutive equations for the heat flux vector $\mathbf{q}$, for the specific internal energy e and for the specific entropy η, and we assume these quantities as functions of

$$p, \theta, \mathbf{v}, \mathbf{L}, \nabla\theta.$$

The Principle of material frame indifference [95] reduces this set of variables to

$$p, \theta, \mathbf{D}, \nabla\theta,$$

and the representation theorems for linear isotropic functions lead us to consider the following constitutive fluid model

$$e = e(p, \theta) + u(p, \theta)\mathrm{tr}\mathbf{D}, \tag{2.19}$$

$$\eta = \eta(p, \theta) + h(p, \theta)\mathrm{tr}\mathbf{D}, \tag{2.20}$$

$$\mathbf{q} = -k(p, \theta)\nabla\theta, \tag{2.21}$$

where k is the *heat conductivity.*

Also the second law of thermodynamics places restrictions on the thermomechanical constitutive equations (2.17), (2.19) and (2.21). To this end we record the second law of thermodynamics in the form of the Clausius-Duhem inequality

$$\rho\dot{\eta} \geq \rho\frac{r}{\theta} - \mathrm{div}\left(\frac{\mathbf{q}}{\theta}\right). \tag{2.22}$$

Inequality (2.22) holds for all themodynamic processes, i.e. for all fields ρ, θ, $\mathbf{v}$ and p satisfying equations (2.5)-(2.7) and (2.9). Hence by Liu Lemma [27, 39, 53] there exist six Lagrange multipliers Λ^ρ, Λ^{v_i} $(i = 1, 2, 3)$, Λ^e and Λ^θ such that, denoting by $d_{<ij>}$ the components of the deviatoric velocity gradient $\mathbf{D} - [(\mathrm{tr}\mathbf{D})/3]\mathbf{1}$,

$$\begin{aligned}
&\rho\left[\frac{\partial\eta}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial\eta}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial\eta}{\partial d_{ii}}\frac{\partial d_{ii}}{\partial t} + v_j\left(\frac{\partial\eta}{\partial p}\frac{\partial p}{\partial x_j} + \frac{\partial\eta}{\partial\theta}\frac{\partial\theta}{\partial x_j} + \frac{\partial\eta}{\partial d_{ii}}\frac{\partial d_{ii}}{\partial x_j}\right)\right] \\
&\quad - \rho\frac{r}{\theta} - \frac{1}{\theta}\frac{\partial k}{\partial p}\frac{\partial p}{\partial x_j}\frac{\partial\theta}{\partial x_j} - \frac{1}{\theta}\left(\frac{\partial k}{\partial\theta} - \frac{k}{\theta}\right)\left(\frac{\partial\theta}{\partial x_j}\right)^2 - \frac{k}{\theta}\frac{\partial^2\theta}{\partial x_j^2} \\
&\quad - \Lambda^\rho\left[\frac{\partial\rho}{\partial t} + v_j\frac{\partial\rho}{\partial x_j} + \rho\frac{\partial v_j}{\partial x_j}\right] \\
&\quad - \Lambda^{v_i}\left[\rho\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right) + \frac{\partial p}{\partial x_i} - 2\left(\frac{\partial\mu}{\partial p}\frac{\partial p}{\partial x_j} + \frac{\partial\mu}{\partial\theta}\frac{\partial\theta}{\partial x_j}\right)d_{<ij>}\right. \\
&\qquad\qquad \left. - 2\mu\frac{\partial d_{<ij>}}{\partial x_j} - \rho b_i\right] \\
&\quad - \Lambda^e\left\{\rho\left[\frac{\partial e}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial e}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial e}{\partial d_{ii}}\frac{\partial d_{ii}}{\partial t} + v_j\left(\frac{\partial e}{\partial p}\frac{\partial p}{\partial x_j} + \frac{\partial e}{\partial\theta}\frac{\partial\theta}{\partial x_j} + \frac{\partial e}{\partial d_{ii}}\frac{\partial d_{ii}}{\partial x_j}\right)\right]\right. \\
&\qquad\qquad \left. - \frac{\partial k}{\partial p}\frac{\partial p}{\partial x_j}\frac{\partial\theta}{\partial x_j} - \frac{\partial k}{\partial\theta}\left(\frac{\partial\theta}{\partial x_j}\right)^2 - k\frac{\partial^2\theta}{\partial x_j^2} + p\frac{\partial v_j}{\partial x_j} - 2\mu d_{<ij>}^2 - \rho r\right\} \\
&\quad - \Lambda^\theta\left[\frac{\partial v_j}{\partial x_j} - \alpha\left(\frac{\partial\theta}{\partial t} + v_j\frac{\partial\theta}{\partial x_j}\right)\right] \geq 0
\end{aligned}$$

for all fields ρ, θ, $\mathbf{v}$ and p, or, equivalently, such that

$$\left.\begin{aligned} \frac{\partial \eta}{\partial p} - \Lambda^e \frac{\partial e}{\partial p} = 0,\\ \frac{\partial \eta}{\partial \theta} - \Lambda^e \frac{\partial e}{\partial \theta} + \frac{\Lambda^\theta}{\rho}\alpha = 0,\\ \frac{\partial \eta}{\partial d_{ii}} - \Lambda^e \frac{\partial e}{\partial d_{ii}} = 0,\\ \Lambda^\rho = 0\\ \Lambda^{v_i} = 0 \quad \forall i = 1,2,3,\\ \Lambda^e = \frac{1}{\theta},\\ \Lambda^\theta + \Lambda^e p = 0,\\ \frac{k}{\theta^2}\left(\frac{\partial \theta}{\partial x_j}\right)^2 + 2\Lambda^e \mu d^2_{<ij>} \geq 0 \quad \text{for all fields} \quad \rho,\ \theta,\ \mathbf{v}\ \text{and}\ p. \end{aligned}\right\} \tag{2.23}$$

By (2.23) we readily deduce that the constitutive functions k and μ are non-negative,

$$\begin{aligned} e &= e(p,\theta), \quad \text{viz } u(p,\theta) = 0 \text{ in (2.19)},\\ \eta &= \eta(p,\theta), \quad \text{viz } h(p,\theta) = 0 \text{ in (2.20)}, \end{aligned}$$

and

$$\frac{\partial e}{\partial p} = \theta \frac{\partial \eta}{\partial p} = -\theta \frac{\alpha}{\rho}, \quad \frac{\partial e}{\partial \theta} = c_p - p\frac{\alpha}{\rho}, \tag{2.24}$$

where $c_p = c_p(p,\theta) = \theta(\partial\eta/\partial\theta)_p$ is the specific heat at constant pressure.

2.3 Governing equations of fluid dynamics

We are now in position to derive from the equations of balance of mass, linear momentum, internal energy and from the constitutive fluid model introduced in the previous section the governing equation of fluid dynamics. We first introduce (2.9) into (2.5) and obtain

$$\frac{\dot\rho}{\rho} = -\alpha(\theta)\dot\theta \tag{2.25}$$

by which we deduce that

$$\alpha = -\frac{1}{\rho}\frac{\partial \rho}{\partial \theta}. \tag{2.26}$$

Next, introducing (2.17), (2.21) and (2.24) into the equations of balance (2.6) and (2.7) gives

$$\rho\dot{\mathbf{v}} = -\nabla p + \frac{\mu}{3}\nabla(\operatorname{div}\mathbf{v}) - \frac{2}{3}(\operatorname{div}\mathbf{v})\nabla\mu + 2\mathbf{D}\cdot\nabla\mu + \mu\Delta\mathbf{v} + \rho\mathbf{b} \tag{2.27}$$

and

$$\rho c_p \dot{\theta} - \alpha\theta\dot{p} = k\Delta\theta + \nabla k \cdot \nabla\theta + 2\mu\left[\|\mathbf{D}\|^2 - \frac{1}{3}(\mathrm{tr}\mathbf{D})^2\right] + \rho r. \qquad (2.28)$$

Equations (2.9), (2.25), (2.27) and (2.28) form the governing equations for the determination of the fields ρ, $\mathbf{v}$, θ and p. It is interesting to note that (2.28) is the equation for the determination of $\dot{p}$ since $\dot{\theta}$ is determined from (2.9).

2.4 Oberbeck-Boussinesq approximation

Few approximations in fluid mechanics have proved as useful and successful in predicting observed phenomena as the Oberbeck-Boussinesq approximation which has implications to a wide variety of flows within the context of astrophysical and geophysical fluid dynamics. The Oberbeck-Boussinesq approximation consists in keeping with a perturbation of the governing equations by identifying a small non-dimensional parameter and retaining terms of like order. While this is the popular wisdom concerning the approximation, this is not a true depiction of the state of affairs as this is not what is strictly carried out in order to obtain the Oberbeck-Boussinesq equations. These celebrated equations are not obtained by a standard perturbation technique. In order to justify the Oberbeck-Boussinesq by appealing to a perturbative approach many arguments have been put forth to justify the inclusion of terms that appear in the equations, but most of these arguments do not pass muster as explained below.

The approximate equations that have been used, and continue to be used, with great success, were first derived by Oberbeck [57, 58] and subsequently and independently derived by Boussinesq [6]. Oberbeck and Boussinesq were interested in obtaining the equations that would govern the flow of a classical linearly viscous fluid which undergoes isochoric motion in isothermal flows, but it could change its volume due to changes in temperature. As we have already seen, this implies that the det $\mathbf{F}$ is a constant in motion when the temperature is a constant, but the value of the det $\mathbf{F}$ could vary with temperature, $\mathbf{F}$ being the deformation gradient. If the motions are sufficiently smooth, this then implies that the div $\mathbf{v}$ vanishes when temperature is a constant but changes when the temperature changes, $\mathbf{v}$ being the velocity of the fluid.

Justification for the approximation due to Oberbeck and Boussinesq are too numerous to be listed and here we mention some of them. Important studies are due to Rayleigh [69], Jeffreys [31], Chandrasekhar [11], Spiegel and Veronis [83], Mihaljan [48], Roberts [76]; Roberts and Stewartson [77], Spiegel and Weiss [84], Hills and Roberts [28], Zeytounian [98]. Not all the above mentioned papers try to provide a rigorous justification for the approximation, some of them do try to provide some sort of rationale for the approximation, but they are not convincing for reasons discussed below. Recently, Rajagopal, Ruzika and Srinivasa [62] carried out an analysis in which they delineate the status of the Oberbeck-Boussinesq approximation based on a certain non-dimensional numbers that they introduce. However, their study implies that the approximation cannot be viewed as a proper perturbation in which terms of like order are retained and in their derivation they show that the Oberbeck-Boussinesq equations result as a consequence of mixing terms of different orders in a small parameter. They also provided higher order approximations to the problem. It might yet be possible to develop a proper perturbation scheme wherein the Oberbeck-Boussinesq equations are obtained as an approximation at a specific order of the perturbation; however at this juncture in time no such analysis is available.

We now discuss briefly some of the attempts to justify the Oberbeck-Boussinesq approximation; a more detailed critique of the various attempts can be found in [62]. Spiegel and Veronis [83] considered the motion of a compressible fluid and they introduced a small parameter ϵ related to the ratio of the variation in density in the absence of motion to the spatial average value of the density and then carried out a perturbation analysis. Spiegel and Veronis were fully aware that their approximation did not retain terms of the same order in the perturbation. In fact, in [83] they explicitly state "In equation (19) we have retained the term $g\epsilon(\rho'/\Delta\rho_0)\mathbf{k}$ even though it contains ϵ as a factor", and this is clearly unacceptable as they recognize. Another shortcoming of the approach of Spiegel and Veronis [83] is that the layer of fluid has to be sufficiently thin while the physical applications, especially in astrophysics and geophysics, require considerably thick layers.

A common problem with many of the justifications for the Oberbeck-Boussinesq approximation stems from the need to retain a term that is the product of the coefficient of thermal expansion and gravity. This product should be of order one, while the coefficient of thermal expansion has to tend to zero. This leads to the untenable requirement that gravity has to tend to infinity. As we saw above, Spiegel and Veronis [83] explicitly retain a term at first order in which the small parameter that appears for the perturbation appears and is multiplied by the acceleration due to gravity. Similarly, in the study by Mihaljan [48] which is often cited for giving a rigorous justification of the Oberbeck-Boussinesq approximation, we encounter a similar difficulty. Mihaljan uses two small parameters for perturbation and he carries out the perturbation analysis. Unfortunately, he does not recognize that when one of the small parameters goes to zero it immediately forces the other small parameter to tend to infinity. In effect he encounters the same problem as that faced by Spiegel and Veronis [83], but under a different guise. Hills and Roberts in their study [28], much in keeping with [83], require that the product of the coefficient of thermal expansion and the acceleration due to gravity be a constant while the coefficient of thermal expansion tends to zero, impossibility if gravity were to be finite. In fact, they recognize the problem with their approach and state explicitly that "As we shall see this last requirement is essential, because otherwise buoyancy forces are lost". Here, the requirement that they refer to is that the product of the coefficient of thermal expansion and the acceleration due to gravity be a constant as the coefficient of thermal expansion tends to zero.

Another attempt at providing a rationale for the Oberbeck-Boussinesq approximation is due to Gray and Giorgini [24]. After providing a very clear discussion of the subtle issues that need to be taken into account in order to obtain the approximation, they make certain ad hoc assumptions concerning the smallness of certain parameters to arrive at the Oberbeck-Boussinesq approximation. Though the study does not provide a rigorous basis for the approximation, their study is an interesting attempt at arriving at the same.

Our study here is similar in its approach as the study by Rajagopal, Ruzika and Srinivasa [62] for the celebrated Oberbeck-Boussinesq equations. However, since the viscosity, the specific heat at constant pressure and the heat conductivity are all functions of both the temperature and pressure and the coefficient of volumetric thermal expansion is temperature dependent, the analysis is much more complicated.

Let us consider a layer of fluid of thickness d, the top and the bottom surfaces of which being held at constant temperature T_2 and T_1 (say $T_1 > T_2$), respectively. In order to non-dimensionalize the equations (2.9), (2.25), (2.27) and (2.28) we choose convenient reference values π_0 and T_0 for pressure and temperature, respectively, and introduce the following dimensionless quantities:

$$\begin{gathered}\mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad \mathbf{v}^* = \frac{\mathbf{v}}{U}, \quad \rho^* = \frac{\rho}{\rho_0}, \quad t^* = \frac{U}{d}t, \\ p^* = \frac{p-\pi_0}{\rho_0 g d}, \quad \mathbf{b}^* = \frac{\mathbf{b}}{g}, \quad \theta^* = \frac{\theta - T_0}{\delta T_0}, \quad \mu^* = \frac{\mu}{\rho_0 U d}, \\ \alpha^* = \frac{\alpha}{\alpha_0}, \quad c_p^* = \frac{\delta T_0}{g d} c_p, \quad k^* = \frac{\delta T_0}{\rho_0 g U d^2} k, \quad r^* = \frac{d}{U^3} r,\end{gathered} \tag{2.29}$$

where

$$\delta T_0 = T_1 - T_2, \quad U = \sqrt{g d \alpha_0 \delta T_0}\,,$$

g is the acceleration due to gravity, ρ_0 and α_0 are the density and the thermal expansion at the reference temperature T_0, respectively. Introducing (2.29) into (2.9), (2.25), (2.27) and (2.28) leads to (omitting all asterisks)

$$\frac{\dot{\rho}}{\rho} = -F^2\alpha\dot{\theta}, \tag{2.30}$$

$$\operatorname{div}\mathbf{v} = F^2\alpha\dot{\theta}, \tag{2.31}$$

$$\begin{aligned} F^2\rho\dot{\mathbf{v}} = &- \nabla p + \frac{F^2}{3}\mu\nabla(\operatorname{div}\mathbf{v}) - \frac{2}{3}F^2(\operatorname{div}\mathbf{v})\nabla\mu \\ &+ 2F^2\nabla\mu\cdot\mathbf{D} + F^2\mu\Delta\mathbf{v} + \rho\mathbf{b} \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} \rho c_p\dot{\theta} - F^2\alpha\left(\theta + \frac{T_0}{\delta T_0}\right)\dot{p} &= k\Delta\theta + \nabla k\cdot\nabla\theta \\ &+ 2F^2\mu\left[\|\mathbf{D}\|^2 - \frac{1}{3}(\mathrm{tr}\mathbf{D})^2\right] + F^2\rho r, \end{aligned} \tag{2.33}$$

where

$$F = \frac{U}{\sqrt{gd}} = \sqrt{\alpha_0\delta T_0}$$

is the Froude number.

We now introduce the small parameter ϵ with respect to which we shall carry out our perturbance. Let

$$\epsilon = F^2 = \frac{U^2}{gd} \ll 1^1$$

and

$$\mathbf{v} = \sum_{n=0}^{+\infty}\epsilon^n\mathbf{v}_n, \quad \theta = \sum_{n=0}^{+\infty}\epsilon^n\theta_n, \quad p = \sum_{n=0}^{+\infty}\epsilon^n p_n \tag{2.34}$$

be the power series in ϵ of the physical quantities $\mathbf{v}$, θ and p. From now on we shall assume that α, c_p, k and μ are analytic functions and we shall limit our analysis to pressure and temperature departures from the reference state (π_0, T_0) for which we can write

$$\alpha(\theta) = \sum_{n=0}^{+\infty}\frac{1}{n!}\frac{\mathrm{d}^n\alpha}{\mathrm{d}\theta^n}(0)\theta^n, \tag{2.35}$$

$$c_p(p,\theta) = \sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}c_p}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p^{j_1}\theta^{j_2}, \tag{2.36}$$

[1]The non-dimensional parameter F^2 is known as the second Froude number.

$$k(p,\theta) = \sum_{j_1+j_2=0}^{+\infty} \frac{1}{j_1!j_2!} \frac{\partial^{(j_1+j_2)}k}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p^{j_1}\theta^{j_2} \tag{2.37}$$

and

$$\mu(p,\theta) = \sum_{j_1+j_2=0}^{+\infty} \frac{1}{j_1!j_2!} \frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)p^{j_1}\theta^{j_2}. \tag{2.38}$$

Thus, from (2.30) and (2.35) we get

$$\rho = \exp\left[-\epsilon\sum_{n=0}^{+\infty}\frac{1}{(n+1)!}\frac{\mathrm{d}^n\alpha}{\mathrm{d}\theta^n}(0)\theta^{n+1}\right]$$

and hence

$$\rho(\theta) = 1-\epsilon\left[\sum_{n=0}^{+\infty}\frac{1}{(n+1)!}\frac{\mathrm{d}^n\alpha}{\mathrm{d}\theta^n}(0)\theta^{n+1}\right]+o(\epsilon), \tag{2.39}$$

where $o(\epsilon)$ represents the terms in ϵ^n with $n \geq 2$.

Inserting (2.34)-(2.39) into (2.31)-(2.33) we get

$$\sum_{n=0}^{+\infty}\epsilon^n \mathrm{div}\, \mathbf{v}_n = \epsilon\sum_{j=0}^{+\infty}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\sum_{n=0}^{+\infty}\epsilon^n\left[\theta^j\left(\frac{\partial\theta_n}{\partial t}+\mathbf{v}\cdot\nabla\theta\right)\right]_n, \tag{2.40}$$

$$\begin{aligned}
&\epsilon\left[1-\epsilon\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\sum_{m=0}^{+\infty}\epsilon^m(\theta^{j+1})_m+o(\epsilon)\right] \\
&\times\sum_{n=0}^{+\infty}\epsilon^n\left[\frac{\partial\mathbf{v}_n}{\partial t}+(\mathbf{v}\cdot\nabla\mathbf{v})_n\right] = -\sum_{n=0}^{+\infty}\epsilon^n\nabla p_n \\
&+\frac{\epsilon}{3}\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left[p^{j_1}\theta^{j_2}\nabla(\mathrm{div}\,\mathbf{v})\right]_n \\
&-\frac{2\epsilon}{3}\sum_{j_1+j_2=1}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left[\mathrm{div}\,\mathbf{v}\nabla(p^{j_1}\theta^{j_2})\right]_n \\
&+2\epsilon\sum_{j_1+j_2=1}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n[\mathbf{D}\cdot\nabla(p^{j_1}\theta^{j_2})]_n \\
&+\epsilon\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left(p^{j_1}\theta^{j_2}\Delta\mathbf{v}\right)_n \\
&+\mathbf{b}\left[1-\epsilon\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\sum_{m=0}^{+\infty}\epsilon^m(\theta^{j+1})_m+o(\epsilon)\right],
\end{aligned} \tag{2.41}$$

$$
\begin{aligned}
&\left[1-\epsilon\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\sum_{m=0}^{+\infty}\epsilon^m(\theta^{j+1})_m+o(\epsilon)\right] \qquad (2.42)\\
&\quad\times\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}c_p}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left[p^{j_1}\theta^{j_2}\left(\frac{\partial\theta}{\partial t}+\mathbf{v}\cdot\nabla\theta\right)\right]_n\\
&\quad-\epsilon\sum_{m=0}^{+\infty}\frac{1}{m!}\frac{\mathrm{d}^m\alpha}{\mathrm{d}\theta}(0)\sum_{n=0}^{+\infty}\left[\theta^m\left(\theta+\frac{T_0}{\delta T_0}\right)\left(\frac{\partial p}{\partial t}+\mathbf{v}\cdot\nabla p\right)\right]_n=\\
&\quad\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}k}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left(p^{j_1}\theta^{j_2}\Delta\theta\right)_n\\
&\quad+\sum_{j_1+j_2=1}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}k}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n[\nabla\theta\cdot\nabla(p^{j_1}\theta^{j_2})]_n\\
&\quad+2\epsilon\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n\left(p^{j_1}\theta^{j_2}\|\mathbf{D}\|^2\right)_n\\
&\quad-\frac{2}{3}\epsilon\sum_{j_1+j_2=0}^{+\infty}\frac{1}{j_1!j_2!}\frac{\partial^{(j_1+j_2)}\mu}{\partial p^{j_1}\partial\theta^{j_2}}(0,0)\sum_{n=0}^{+\infty}\epsilon^n[p^{j_1}\theta^{j_2}(\mathrm{tr}\mathbf{D})^2]_n\\
&\quad+\epsilon r\left[1-\epsilon\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\sum_{m=0}^{+\infty}\epsilon^m(\theta^{j+1})_m+o(\epsilon)\right].
\end{aligned}
$$

We are now in position to equate the like powers of ϵ and obtain a sistematic hierarchy of equations. Collecting the terms of $O(1)$ in equations (2.40)-(2.42) we obtain

$$\mathrm{div}\,\mathbf{v}_0=0, \qquad (2.43)$$

$$-\nabla p_0+\mathbf{b}=\mathbf{0} \qquad (2.44)$$

and

$$c_p(p_0,\theta_0)\left(\frac{\partial\theta_0}{\partial t}+\mathbf{v}_0\cdot\nabla\theta_0\right)=k(p_0,\theta_0)\Delta\theta_0+\nabla[k(p_0,\theta_0)]\cdot\nabla\theta_0. \qquad (2.45)$$

We notice that the above equations are not sufficient to determine all the field variables at $O(1)$. Therefore, in order to attain closure, we proceed to obtain the equations at $O(\epsilon)$. Setting

$$G(\theta_0)=\int_0^{\theta_0}\alpha(\theta)\mathrm{d}\theta=\sum_{j=0}^{+\infty}\frac{1}{(j+1)!}\frac{\mathrm{d}^j\alpha}{\mathrm{d}\theta^j}(0)\theta_0^{j+1},$$

from (2.41) we obtain

$$\frac{\partial \mathbf{v}_0}{\partial t}+\mathbf{v}_0\cdot\nabla\mathbf{v}_0=-\nabla p_1+\frac{1}{3}\mu(p_0,\theta_0)\nabla(\mathrm{div}\,\mathbf{v}_0)-\frac{2}{3}(\mathrm{div}\,\mathbf{v}_0)\nabla[\mu(p_0,\theta_0)]$$
$$+\,2\mathbf{D}_0\cdot\nabla[\mu(p_0,\theta_0)]+\mu(p_0,\theta_0)\Delta\mathbf{v}_0-G(\theta_0)\mathbf{b}$$

which, in the light of (2.43), becomes

$$\frac{\partial \mathbf{v}_0}{\partial t}+\mathbf{v}_0\cdot\nabla\mathbf{v}_0=-\nabla p_1+2\mathbf{D}_0\cdot\nabla[\mu(p_0,\theta_0)]+\mu(p_0,\theta_0)\Delta\mathbf{v}_0-G(\theta_0)\mathbf{b}. \tag{2.46}$$

Now equations (2.43)-(2.46) form a closed system and it is interesting to remark that p_0 is the pressure due to the body forces acting on the fluid while p_1 is the pressure due to the thermal expansion of the fluid. Next, by means of (2.29) we re-dimensionalize equations (2.43)-(2.46) and obtain the equations governing the flows in a fluid layer at small second Froude number

$$\begin{cases} -\nabla p_0+\rho_0\mathbf{b}=\mathbf{0} \\ \rho_0\left(\dfrac{\partial \mathbf{v}_0}{\partial t}+\mathbf{v}_0\cdot\nabla\mathbf{v}_0\right)=-\alpha_0(T_1-T_2)\nabla p_1 \\ \qquad +2\mathbf{D}_0\cdot\nabla[\mu(p_0,\theta_0)]+\mu(p_0,\theta_0)\Delta\mathbf{v}_0-\rho_0 G(\theta_0)\mathbf{b} \\ \mathrm{div}\,\mathbf{v}_0=0 \\ \rho_0 c_p(p_0,\theta_0)\left(\dfrac{\partial \theta_0}{\partial t}+\mathbf{v}_0\cdot\nabla\theta_0\right)=k(p_0,\theta_0)\Delta\theta_0 \\ \qquad +\nabla[k(p_0,\theta_0)]\cdot\nabla\theta_0, \end{cases} \tag{2.47}$$

where the function G is now defined as

$$G(\theta_0)=\int_{T_0}^{\theta_0}\alpha(\theta)\mathrm{d}\theta.$$

It is easy to check that, if α, c_p, k and μ are assumed to depend only on temperature, system (2.47) simplifies to

$$\begin{cases} \rho_0\left(\dfrac{\partial \mathbf{v}_0}{\partial t}+\mathbf{v}_0\cdot\nabla\mathbf{v}_0\right)=-\nabla p+2\mathbf{D}_0\cdot\nabla\mu(\theta_0)+\mu(\theta_0)\Delta\mathbf{v}_0-\rho\mathbf{b} \\ \mathrm{div}\,\mathbf{v}_0=0 \\ \rho_0 c_p(\theta_0)\left(\dfrac{\partial \theta_0}{\partial t}+\mathbf{v}_0\cdot\nabla\theta_0\right)=k(\theta_0)\Delta\theta_0+\nabla k(\theta_0)\cdot\nabla\theta_0, \end{cases} \tag{2.48}$$

where

$$p=p_0+\alpha_0(T_1-T_2)p_1 \quad \text{and} \quad \rho=\rho_0[1-G(\theta_0)].$$

Finally, if α, c_p, k and μ are supposed to be constant, (2.48) reduces to the classical Oberbeck-Boussinesq equations [11, 17, 87]

$$\begin{cases} \rho_0\left(\dfrac{\partial \mathbf{v}_0}{\partial t}+\mathbf{v}_0\cdot\nabla\mathbf{v}_0\right)=-\nabla p+\mu\Delta\mathbf{v}_0-\rho\mathbf{b} \\ \mathrm{div}\,\mathbf{v}_0=0 \\ \dfrac{\partial \theta_0}{\partial t}+\mathbf{v}_0\cdot\nabla\theta_0=\kappa\Delta\theta_0, \end{cases}$$

where $\rho = \rho_0[1 - \alpha(\theta_0 - T_0)]$ and $\kappa = k/(\rho_0 c_p)$ is the *thermal diffusivity.*

2.5 Equations of magnetohydrodynamics

The objective of magnehydrodynamics (MHD) is the study of the ways in which magnetic fields can affect the behaviour of electrically conducting fluids. The electrical conductivity of the fluid and the embedding magnetic field contibute to effects of two kinds. First, as the electrically conducting fluid moves across the magnetic lines of force, electric currents are generated in the fluid (according to Faraday-Neumann-Lenz law) and the induced magnetic field contributes to change in the existing field. At the same time the fluid elements carrying currents transverse magnetic lines of force contribute to additional forces (Lorentz forces) which modify the motion and to additional supplies to internal energy due to Joule effect.

The equations governing the interactions between the electromagnetic field and the motion of an electrically conducting fluid are based upon the assumption of validity of Maxwell's equations. Since changes in time of electric and magnetic fields ($\mathbf{E}$ and $\mathbf{H}$, respectively) are determined by the instantaneous distribution of $\mathbf{E}$ and $\mathbf{H}$ and by the motion of the electric charges, irrespective of how this distribution and this motion are produced, Maxwell's equations are not formally altered by the fluid motion. Then, denoting by ϵ_e the dielectric constant of the fluid and by μ_e the magnetic permeability, we have

$$\operatorname{curl} \mathbf{H} = \mathbf{J} + \mathbf{D}_t\,, \tag{2.49}$$

$$\operatorname{curl} \mathbf{E} = -\mathbf{B}_t\,, \tag{2.50}$$

$$\operatorname{div} \mathbf{B} = 0\,, \tag{2.51}$$

$$\operatorname{div} \mathbf{D} = \rho_e\,, \tag{2.52}$$

$$\mathbf{B} = \mu_e \mathbf{H} \quad \text{and} \quad \mathbf{D} = \epsilon_e \mathbf{E}, \tag{2.53}$$

where the vectors $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{J}$ are, respectively, the *magnetic induction*, the *electric induction* (or *displacement vector*) and the *current density*, $\mathbf{D}_t$ is the *displacement current* and the scalar quantity ρ_e represents the *electric charge density.* The current density $\mathbf{J}$, expressed through Ohm's law, is the sum of the *conduction current*

$$\sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

and the *convection current* $\rho_e \mathbf{v}$. The equation for the current density is therefore

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) + \rho_e \mathbf{v} \tag{2.54}$$

which is to be added to equations (2.49)-(2.53).

The governing equations of magnetohydrodynamics are obtained by coupling the equations of electromagnetism (2.49)-(2.54) and the equations of fluid dynamics (2.5), (2.27) and (2.28), the last two ones containing additional terms due to the interactions between the fluid and the electromagnetic field, viz

$$\rho\dot{\mathbf{v}} = -\nabla p + \frac{\mu}{3}\nabla(\operatorname{div}\mathbf{v}) - \frac{2}{3}(\operatorname{div}\mathbf{v})\nabla\mu + 2\mathbf{D}\cdot\nabla\mu + \mu\Delta\mathbf{v} + \rho\mathbf{b} + \mathbf{J}\times\mathbf{B} \quad (2.55)$$

and

$$\rho c_p\dot{\theta} - \alpha\theta\dot{p} = k\Delta\theta + \nabla k\cdot\nabla\theta + 2\mu\left[\|\mathbf{D}\|^2 - \frac{1}{3}(\operatorname{tr}\mathbf{D})^2\right] + \rho r + \frac{|\mathbf{J}|^2}{\sigma}. \quad (2.56)$$

$\mathbf{J}\times\mathbf{B}$ is the Lorentz force and $|\mathbf{J}|^2/\sigma$ is the heat produced by Joule effect.

Equations (2.5), (2.55) and (2.56) are invariant with respect to Galileian transformations whereas Maxwell's equations are invariant with respect to Lorentz's transformations. Thus, in order to obtain a coherent system of PDEs, as in most problems involving conductors, other than those concerned with rapid oscillations, the displacement current can be ignored so that, as it is well kown, also Maxwell's equations are invariant with respect to Galileian transformations (see [12]).

Let now L, t_0, V, E_0 and H_0 be typical values of lenght, time, velocity, electric and magnetic fields, respectively, and, by following Agostinelli [1], let us assume that

$$R_t = \frac{t_0 V}{L} \simeq 1, \quad (2.57)$$

$$R_e = \frac{E_0}{\mu_e H_0 V} \simeq 1 \quad (2.58)$$

and

$$\frac{V^2}{c^2} = V^2\epsilon_e\mu_e \ll 1. \quad (2.59)$$

By assumption (2.57) we do not consider high frequency phenomena. Condition (2.58) is a good approximation for fluids having a very large electric conductivity because, for $\sigma\to\infty$, from (2.54) one has $\mathbf{E} = -\mu_e\mathbf{v}\times\mathbf{H}$ and hence $|\mathbf{J}|$ is of order $\sigma\mu_e H_0 V$. Finally by (2.59) we assume that the fluid velocity is much smaller than the light speed in the fluid. As consequences of (2.57)-(2.59) we shall see that the displacement and the convection currents can be neglected. We first introduce the following scaling

$$\begin{aligned} &\mathbf{x}^* = \frac{\mathbf{x}}{L}, \quad t^* = \frac{t}{t_0}, \quad \mathbf{v}^* = \frac{\mathbf{v}}{V} \\ &\mathbf{E}^* = \frac{\mathbf{E}}{E_0}, \quad \mathbf{H}^* = \frac{\mathbf{H}}{H_0}, \quad \mathbf{J}^* = \frac{\mathbf{J}}{\mu_e\sigma V H_0}, \end{aligned} \quad (2.60)$$

and the magnetic Reynolds number

$$R_m = \frac{VL}{\eta},$$

$\eta = (\mu_e \sigma)^{-1}$ being the *magnetic visosity* (or *magnetic diffusivity*). Then, introducing the non dimensional quantities (2.60) into equation (2.49) yields (omitting all asterisks)

$$\frac{1}{R_m}\mathrm{curl}\,\mathbf{H} = \mathbf{J} + \frac{R_c R_e}{R_t R_m}\mathbf{E}_t.$$

By assumptions (2.57)-(2.59) we can thus ignore the displacement current and so (2.49) becomes

$$\mathrm{curl}\,\mathbf{H} = \mathbf{J}. \tag{2.61}$$

In a similar way, writing equation (2.54) by taking into account (2.52) and $(2.53)_2$, the dimensionless equation for the current density is

$$\mathbf{J} = (R_e\mathbf{E} + \mathbf{v}\times\mathbf{H}) + \frac{R_c R_e}{R_m}(\mathrm{div}\,\mathbf{E})\mathbf{v}.$$

This equation shows that, since $\frac{R_c R_e}{R_m} \ll 1$, the convective current is negligible with respect to the conduction current, and so

$$\mathbf{J} = \sigma(\mathbf{E} + \mu_e\mathbf{v}\times\mathbf{H}). \tag{2.62}$$

We now observe that, by taking the curl of both sides of equation (2.61), by means of equations (2.50), (2.51), $(2.53)_1$ and (2.62) we get

$$\mathbf{H}_t + \mathrm{curl}\,(\mathbf{H}\times\mathbf{v}) = \eta\Delta\mathbf{H}.$$

Moreover, by (2.62) and $(2.53)_1$, the Lorentz force is given by

$$\mathbf{J}\times\mathbf{B} = \mu_e\mathrm{curl}\,\mathbf{H}\times\mathbf{H} = \mu_e\mathbf{H}\cdot\nabla\mathbf{H} - \frac{\mu_e}{2}\nabla|\mathbf{H}|^2,$$

$(\mu_e\nabla|\mathbf{H}|^2)/2$ being the *magnetic pressure*, and the heat produced by Joule effect is

$$\frac{|\mathbf{J}|^2}{\sigma} = \frac{|\mathrm{curl}\,\mathbf{H}|^2}{\sigma}.$$

Therefore the governing equations of non relativistic MHD are

$$
\begin{cases}
\dot{\rho} + \rho \mathrm{div}\, \mathbf{v} = 0 \\
\rho\dot{\mathbf{v}} = -\nabla\left(p + \frac{\mu_e}{2}\nabla|\mathbf{H}|^2\right) + \frac{\mu}{3}\nabla(\mathrm{div}\, \mathbf{v}) - \frac{2}{3}(\mathrm{div}\, \mathbf{v})\nabla\mu \\
\qquad\qquad +2\mathbf{D}\cdot\nabla\mu + \mu\Delta\mathbf{v} + \rho\mathbf{b} + \mu_e\mathbf{H}\cdot\nabla\mathbf{H} \\
\rho c_p\dot{\theta} - \alpha\theta\dot{p} = k\Delta\theta + \nabla k\cdot\nabla\theta + 2\mu\left[\|\mathbf{D}\|^2 - \frac{1}{3}\left(\mathrm{tr}\mathbf{D}\right)^2\right] \\
\qquad\qquad\qquad +\rho r + \dfrac{|\mathrm{curl}\, \mathbf{H}|^2}{\sigma} \\
\mathbf{H}_t + \mathrm{curl}\, (\mathbf{H}\times\mathbf{v}) = \eta\Delta\mathbf{H} \\
\mathrm{div}\, \mathbf{H} = 0
\end{cases}
\tag{2.63}
$$

and form a coherent system of PDEs.

2.6 Porous media

By a *porous medium* we mean a material consisting of a solid matrix with interconnected void. We suppose that the solid matrix is rigid. The interconnectedness of the void (the *pores*) allows the flow of one or more fluids through the material. In the simplest situation (the single-phase flow) the void is saturated by a single fluid. In two-phase flow a liquid and a gas share the void space. Here we shall discuss the former situation.

In a natural porous medium the distribution of pores with respect to shape and size is irregular. Examples of natural porous media are beach sand, sandstone, limestone, wood and human lung. On the pore scales (the microscopic scale) the flow quantities (velocity, pressure, etc.) will clearly be irregular. But in typical experiments the quantities of interest are measured over volumes that contain many pores. Such space-averaged (macroscopic) quantities change in a regular manner with respect to space and time and hence are amenable to theoretical treatment.

The usual way of deriving the laws governing the macroscopic variables is to begin with the standard equations obeyed by the fluid and to obtain the macroscopic equations by averaging over volumes containing many pores. In this approach, a macroscopic variable is defined as an appropriate mean over a sufficiently large *representative elementary volume* (r.e.v.); this operation yields the value of that variable at the centre of the r.e.v.. It is assumed that the result is independent of the size of the representative elementary volume. The lenght scale of r.e.v. is much larger than the pore scale, but considerably smaller than the lenght scale of the macroscopic flow domain.

2.7 Porosity, seepage velocity and the equation of continuity

The porosity φ of a porous medium is defined as the fraction of the total volume of the medium that is occupied by void space, that is

$$\varphi = \frac{\text{total volume of the pores}}{\text{total volume of the medium}}.$$

Thus $1 - \varphi$ is the fraction that is occupied by the solid. For an isotropic medium the *surface porosity* (i.e. the fraction of void area to total area of a typical cross section) will normally be equal to φ.

For natural media, φ does not normally exceed 0.6. Nonuniformity of grain size tends to lead to smaller porosities than for uniform grains. For man-made materials such as metallic foams φ can approach the value 1.

We construct a continuum model for a porous medium based on the r.e.v. concept. We introduce a Cartesian frame of reference and consider volume elements that are sufficiently large compared with the pore volumes in order to obtain reliable volume averages. In other words, the averages are not sensitive to the choice of volume element. A distinction is made between an average taken with respect to a volume element V_m (incorporating both solid and fluid material), and one taken with respect to a volume V_f consisting of fluid only. For example, we denote the average of fluid velocity over V_m by $\mathbf{v}$ which is usually called the *seepage velocity.* Taking an average of the fluid velocity over a volume V_f we get the intrinsic average velocity $\mathbf{V}$, which is related to $\mathbf{v}$ by the Dupuit-Forchheimer relationship

$$\mathbf{v} = \varphi \mathbf{V}. \tag{2.64}$$

Once we have a continuum to deal with, we can apply the usual arguments of section 2.1 to derive differential equations expressing conservation laws. For instance, denoting by ρ_f the fluid density and considering an elementary unit volume of the medium V, the conservation of mass is expressed by

$$\begin{aligned} 0 = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \varphi\rho_f \mathrm{d}V &= \int_V \left[\frac{\partial(\varphi\rho_f)}{\partial t} + \mathrm{div}(\varphi\rho_f\mathbf{V})\right]\mathrm{d}V \qquad (2.65)\\ &= \int_V \left[\varphi\frac{\partial \rho_f}{\partial t} + \mathrm{div}(\rho_f\mathbf{v})\right]\mathrm{d}V, \end{aligned}$$

where we have taken into account (2.64) and that φ is independent of t. By (2.65) we then deduce the continuity equation in a porous medium

$$\varphi\frac{\partial \rho_f}{\partial t} + \mathrm{div}(\rho_f\mathbf{v}) = 0. \tag{2.66}$$

2.8 Linear momentum equation in a porous medium

Following the same arguments that lead to the equation of continuity in a porous medium, we shall derive the most general equation of balance of linear momentum when the porous medium is isotropic and homogeneous, i.e. the porosity φ is constant. Let V be an elementary unit volume of the medium and equate the rate of change of the linear momentum of the fluid within that volume to the net forces acting on the fluid into the volume V:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varphi\rho_f \mathbf{V}\mathrm{d}V = \int_{\partial V} \varphi\mathbf{T}\cdot\mathbf{n}\mathrm{d}A + \int_V \varphi\rho_f \mathbf{b}\mathrm{d}V + \int_V \varphi\mathbf{I}\mathrm{d}V, \tag{2.67}$$

where

$$\mathbf{T} = -p\mathbf{1} + 2\mu_f\left[\left(\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\right) + \left(\frac{\partial\mathbf{V}}{\partial\mathbf{x}}\right)^{\mathrm{T}}\right]$$

is the stress tensor in the fluid, μ_f being the fluid viscosity which, for simplicity, is now assumed to be a constant, $\mathbf{b}$ is the body force and $\mathbf{I}$ is the density of interaction forces between the fluid and the porous matrix. By Reynolds' transport Theorem (2.1), the arbitrariety of the volume V, the Dupuit-Forchheimer relationship (2.64) and the equation of continuity (2.66), (2.67) yields the local balance of momentum

$$\frac{\rho_f}{\varphi}\mathbf{v}_t + \frac{\rho_f}{\varphi^2}\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p + \frac{\mu_f}{\varphi}\left[\Delta\mathbf{v} + \nabla(\mathrm{div}\,\mathbf{v})\right] + \mathbf{I} + \rho_f\mathbf{b}. \tag{2.68}$$

We now discuss various approximated forms of the momentum equation (2.68) and the basic assumptions which justify them.

2.8.1 Darcy's law

Darcy's investigations on steady-state flow in a uniform porous medium [13] revealed an equation for the linear momentum of the type

$$\nabla p = -\frac{\mu}{\varphi K}\mathbf{v} + \rho_f\mathbf{b}, \tag{2.69}$$

where μ is the *dynamic viscosity* of the fluid and the coefficient K is independent of the nature of the fluid but it depends on the geometry of the medium. K has dimensions (lenght)^2 and is called the *permeability* of the medium.

Following Rajagopal [66], the basic assumptions leading to (2.69) are that:

(i) The solid is a rigid porous body and thus the balance of the linear momentum of the solid can be ignored.

(ii) The only interaction forces that come into play are due to frictional forces the fluid encounters at the boundaries of the pores. This can be modelled by a drag term proportional to the fluid velocity. The coefficient of proportionality being a constant.

(iii) The frictional effects within the fluid due to its viscosity can be neglected.

(iv) The flow is sufficiently slow that the inertial nonlinearities can be neglected.

(v) The flow is steady.

(vi) The density of the fluid is constant.

(vii) The stress for the fluid is that for an ideal Euler fluid as the frictional effects in the fluid can be neglected with respect to the frictional effects in the pore (which has already been incorporated in the interaction term).

Assumption (i) implies that we need to concern ourselves with only the balance of linear momentum for the fluid as the porous matrix is rigid and and does not deform. Thus on fixing the frame to the porous matrix the velocity of the solid is zero. Next, assumption (ii) implies that

$$\mathbf{I} = -\frac{\mu}{K}\mathbf{V},$$

where the *dynamic viscosity* μ is usually assumed to be a constant.

Assumptions (iv) and (v) imply that the inertial terms in the right-hand side of (2.68) can be ignored.

Assumption (iii) implies that as far as the response of the fluid is concerned, the effects of viscosity (frictional effects) can be neglected with respect to the friction that manifests itself due to the flow in the pores. This does not mean that the fluid has no viscosity. In fact, assumptions (ii) and (iii) together imply that the viscosity of the fluid and the roughness of the solid surface lead to far greater frictional resistance at the porous boundaries of the solid in comparison to the frictional dissipation in the fluid, but these two assumptions do not necessarily imply that the fluid stress tensor is that for an Euler fluid. Only by assumption (vii) we can approximate the Cauchy stress tensor of the fluid as $\mathbf{T} = -p\mathbf{1}$.

Finally, since ρ_f is constant, the fluid can undergo only isochoric motions and the equation of continuity (2.66) reduces to

$$\operatorname{div} \mathbf{v} = 0. \tag{2.70}$$

Equations (2.69) and (2.70) constitute what is referred as Darcy's law. The subsequent generalizations of (2.69) [56] (such as that carried out by Forchheimer [21]) can be easily obtained by modifying the form of the interaction term.

2.8.2 Brinkman's equations

Let us now relax some of the assumptions (*i*)-(*vii*). We shall not enforce the assumptions (*iii*) and (*vii*) while we shall retain the other ones. We shall then include the frictional forces in the fluid when we consider the balance of linear momentum. The equation of balance of linear momentum (2.68) then becomes

$$-\nabla p + \frac{\mu_f}{\varphi}\Delta \mathbf{v} - \frac{\mu}{\varphi K}\mathbf{v} + \rho_f \mathbf{b} = \mathbf{0}. \tag{2.71}$$

Let us observe that whenever the lenght scale is much greater than $(\mu_f K/\mu)^{1/2}$, the Laplacian term in equation (2.71) is negligible in comparison with the term proportional to $\mathbf{v}$ so that the Brinkman's equation reduce to Darcy's equation. In fact, if we introduce the following dimensionless quantities

$$\mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad \mathbf{b}^* = \frac{\mathbf{b}}{g}, \quad \mathbf{v}^* = \frac{\mu}{\rho_f g K}\mathbf{v}, \quad p^* = \frac{p}{\rho_f g d}, \tag{2.72}$$

where d is the lenght scale of the porous medium and g is the acceleration due to gravity, and substitute (2.72) into (2.71), we obtain the non-dimensional Brinkman's equation (omitting all asterisks)

$$-\nabla p + \frac{\mu_f K}{\mu d^2}\Delta \mathbf{v} - \mathbf{v} + \mathbf{b} = \mathbf{0}. \tag{2.73}$$

Therefore if

$$\frac{\mu_f K}{\mu d^2} \ll 1$$

(2.73) reduces to the dimensionless version of (2.69).

If we do not require the flow to be steady but assume that it is sufficiently slow that inertial nonlinearities can be neglected we get the unsteady Brinkmann's equation

$$\frac{\rho_f}{\varphi}\mathbf{v}_t = -\nabla p + \frac{\mu_f}{\varphi}\Delta \mathbf{v} - \frac{\mu}{\varphi K}\mathbf{v} + \rho_f \mathbf{b}.$$

Neglecting the frictional effects in the fluid, the above equation will lead to the unsteady version of Darcy's equation.

2.9 Energy equation in a porous medium

We now focus on the equation that express the balance of internal energy in a porous medium. We concentrate our attention on the simplest situation in which the medium is isotropic, homogeneous and where radiative effects, viscous dissipation and the work done by the pressure changes are negligible. Very shortly we shall assume that there is local equilibrium so that $T_s = T_f = T$, where T_s and T_f are the temperature of the solid matrix and of the fluid, respectively. Moreover we also assume that heat conduction in the porous matrix and in the fluid takes place in parallel so that there is no net heat transfer from one constituent to the other. More complex situations are considered in the book of Nield and Bejan Chapter 2 and Section 6.5.

Taking averages over an elemental volume of the medium we have, for the solid matrix,

$$(1-\varphi)(\rho c)_s \frac{\partial T_s}{\partial t} = (1-\varphi)\mathrm{div}(k_s \nabla T_s) \tag{2.74}$$

and, for the fluid,

$$\varphi(\rho c_p)_f \frac{\partial T_f}{\partial t} + (\rho c_p)_f \mathbf{v}\cdot\nabla T_f = \varphi \mathrm{div}(k_f \nabla T_f). \tag{2.75}$$

Here the subscripts s and f refer to the solid matrix and to the fluid, respectively, c is the specific heat of the solid, c_p is the specific heat at constant pressure of the fluid and k is the heat conductivity.

In writing equations (2.74) and (2.75) we have assumed that the surface porosity is equal to the porosity. This is pertinent to the conduction terms. For instance, $-k_s\nabla T_s$ is the conductive heat flux through the solid and thus $\mathrm{div}(k_s\nabla T_s)$ is the net rate of heat conduction into a unit volume of the solid. In equation (2.74) this appears multiplied by the factor $1-\varphi$ which is the ratio of the cross-sectional area of the medium. The other term in equation (2.74) contains the factor $1-\varphi$ because this is the ratio of the volume occupied by the solid to the total volume of the element. In equation (2.75) there also appears a convective term, due to the seepage velocity. We recognize that $\mathbf{V}\cdot\nabla T_f$ is the rate of change of temperature in the elemental volume due to the convection of the fluid into it, so this, multiplied by $(\rho c_p)_f$, must be the rate of change of thermal energy, per unit volume of the fluid, due to the convection. Note that in writing equation (2.75) we have used the Dupuit-Forchheimer relationship (2.64).

Setting $T_s = T_f = T$ and adding equations (2.74) and (2.75) we have

$$(\rho c)_m \frac{\partial T}{\partial t} + (\rho c_p)_f \mathbf{v}\cdot\nabla T = \mathrm{div}(k_m \nabla T), \tag{2.76}$$

where

$$(\rho c)_m = (1-\varphi)(\rho c)_s + \varphi(\rho c_p)_f$$

and

$$k_m = (1-\varphi)k_s + \varphi k_f$$

are, respectively, the overall heat capacity per unit volume and the overall thermal conductivity of the medium.

If the work done by the pressure changes is not negligible, then a term $-\alpha_f T(\varphi \partial p/\partial t + \mathbf{v}\cdot\nabla p)$ needs to be added to the left hand side of equation (2.76). Here α_f is the coefficient of volumetric thermal expansion of the fluid defined in (2.26). In natural convection the work done by the pressure changes is negligible if

$$\frac{g\alpha_f d}{c_{pf}} \ll 1, \tag{2.77}$$

d being a characteristic lenght scale of the medium, as one can easiliy deduce from the non-dimensional analysis performed in section 2.4. In natural convection the condition (2.77) is usually verified.

Chapter 3

Laminar flows in fluids with tempearture and pressure dependent viscosity

3.1 Introduction

In this chapter and in the next one we shall consider fluids whose viscosity is an analytic function of both temperature and pressure but its coefficient of thermal expansion α, its thermal conductivity k and its specific heat at constant pressure c_p are constants. While it is true that all the physical quantities do vary with pressure, the variation in the viscosity with pressure is far more dramatic than the variation of the other quantities with pressure. We shall now use the Barus' equation (2.11) to get a rough estimate of the variation in the viscosity with pressure for common organic liquids. For Naphthalemic mineral oil the piezoviscous coefficient β has been determined experimentally to be 26.5 GPa^{-1} at 20 °C, 23.4 GPa^{-1} at 40 °C, 20 GPa^{-1} at 60 °C and 16.4 GPa^{-1} at 80 °C (see [29] for details). Thus a change of pressure from 0.1 GPa to 1.0 GPa at 80 °C leads to a change in the viscosity of $2.57 \cdot 10^8$%! The density on the other hand changes according to the relation [16]

$$\rho = \rho_0 \left(1 + \frac{0.6p}{1 + 1.7p}\right),$$

and thus, the change in density is merely 16%. While such a change in density will be taken into account if one is interested in depicting the response very accurately, in most applications one can ignore the density change and model the fluid as incompressible. The other properties also undergo much more modest changes in their values than the viscosity and hence we feel that assuming α, k and c_p constants is a reasonable first approximation.

Based on this approximation we study steady unidirectional flows subject to temperature field to assess the effect of buoyancy on the flow when the viscosity depends upon the pressure. In particular we study the laminar flows in polymer melts (section 3.3) and in bitumen (section 3.4).

3.2 Laminar flows

Let $Oxyz$ be a cartesian frame of reference with unit vector fields $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively, $\mathbf{k}$ pointed vertically upward. In this section we shall determine the laminar flows in a fluid whose viscosity is an analytic function of temperature and pressure whereas the coefficient of thermal expansion α, the specific heat at constant pressure c_p and the heat conductivity k are assumed to be constants. Therefore, if gravity is the only force acting on the fluid, the equations which govern the motion we have derived in section 2.4 become:

$$\begin{cases} \nabla p + \rho_0 g \mathbf{k} = 0 \\ \rho_0 \mathbf{v}_t + \rho_0 \mathbf{v} \cdot \nabla \mathbf{v} = -\alpha(T_1 - T_2)\nabla P + \mu(p,T)\Delta \mathbf{v} \\ \qquad\qquad +2\mathbf{D} \cdot \nabla \mu(p,T) + \rho_0 g \alpha (T - T_0)\mathbf{k} \\ \operatorname{div} \mathbf{v} = 0 \\ T_t + \mathbf{v} \cdot \nabla T = \kappa \Delta T \end{cases} \tag{3.1}$$

in $\Omega_d = \mathbb{R}^2 \times (-d/2, d/2)$. In (3.1) ρ_0 is the density at the reference temperature $T_0 = (T_1 + T_2)/2$, $\kappa = k/(\rho_0 c_p)$ is the thermal diffusivity, g and p are, respectively, the acceleration and the pressure field due to gravity, P is the pressure due to the thermal expansion of the fluid and by T we denote the temperature field. The boundary conditions we append to system (3.1) are

$$\begin{cases} T(x,y,d/2,t) = T_2, \quad T(x,y,-d/2,t) = T_1 \\ p(x,y,0,t) = p_0 \end{cases} \tag{3.2}$$

where p_0 is the reference pressure.

Now it is convenient to non-dimensionalize (3.1) according to the scales:

$$\begin{aligned} &\mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad t^* = \frac{\mu_0}{\rho_0 d^2} t, \quad \mathbf{v}^* = \frac{\rho_0 d}{\mu_0}\mathbf{v}, \\ &p^* = \frac{p - p_0}{\rho_0 g d}, \quad P^* = \frac{P}{\rho_0 g d}, \quad \mu^* = \frac{\mu}{\mu_0}, \\ &T^* = \frac{T - T_0}{T_1 - T_2}, \quad R = \frac{\alpha(T_1 - T_2)\rho_0 g d^3}{\mu_0 \kappa}, \quad Pr = \frac{\mu_0}{\rho_0 \kappa}, \end{aligned} \tag{3.3}$$

where $\mu_0 = \mu(p_0, T_0)$ is the viscosity at the reference state (p_0, T_0), R and Pr are the Rayleigh and Prandtl numbers, respectively. With this scaling

(3.1) becomes (omitting all asterisks)

$$\begin{cases} \nabla p + \mathbf{k} = 0 \\ \mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{R}{Pr} \nabla P + \mu(p,T) \Delta \mathbf{v} + 2\mathbf{D} \cdot \nabla \mu(p,T) + \frac{R}{Pr} T \mathbf{k} \\ \operatorname{div} \mathbf{v} = 0 \\ Pr(T_t + \mathbf{v} \cdot \nabla T) = \Delta T \end{cases} \tag{3.4}$$

in $\mathbb{R}^2 \times (-1/2, 1/2)$. Then to determine the steady flows of the type

$$\mathbf{v} = v(z)\mathbf{i}, \quad T = T(z),$$

we have to solve the following system

$$\begin{cases} p_x = p_y = P_y = 0 \\ p_z = -1 \\ -\frac{R}{Pr} P_x + \mu_z v_z + \mu v_{zz} = 0 \\ -\frac{R}{Pr} P_z + \mu_x v_z + \frac{R}{Pr} T = 0 \\ T_{zz} = 0 \end{cases} \tag{3.5}$$

with boundary conditions

$$\begin{cases} v(-1/2) = V_1, \quad v(1/2) = V_2 \\ T(-1/2) = 1/2, \quad T(1/2) = -1/2 \\ p(0) = 0. \end{cases} \tag{3.6}$$

It is easy to check that the boundary value problem (3.5)-(3.6) admits the solution

$$\begin{cases} p = T = -z \\ P = -\frac{z^2}{2} + \frac{Pr}{R} A_0 x + P_0 \\ v = V_1 + \int_{-1/2}^{z} \frac{A_0 \zeta + c}{\mu(\zeta)} \mathrm{d}\zeta, \end{cases} \tag{3.7}$$

where A_0 is the pressure gradient and

$$c = \left[V_2 - V_1 - A_0 \int_{-1/2}^{1/2} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta \right] \left[\int_{-1/2}^{1/2} \frac{\mathrm{d}\zeta}{\mu(\zeta)} \right]^{-1}.$$

We have therefore a one-parameter family of laminar flows, the pressure gradient A_0 being the variable parameter, which includes two important special cases:

- for $A_0 = 0$, $V_2 = V$ and $V_1 = -V$, the *Couette flow*

$$v = -V + \frac{2V}{\displaystyle\int_{-1/2}^{1/2} \frac{\mathrm{d}\zeta}{\mu(\zeta)}} \int_{-1/2}^{z} \frac{\mathrm{d}\zeta}{\mu(\zeta)}; \tag{3.8}$$

- for $A_0 \neq 0$ and $V_1 = V_2 = 0$, the *Poiseuille flow*

$$v = A_0 \left[\int_{-1/2}^{z} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta - \frac{\displaystyle\int_{-1/2}^{1/2} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta}{\displaystyle\int_{-1/2}^{1/2} \frac{\mathrm{d}\zeta}{\mu(\zeta)}} \int_{-1/2}^{z} \frac{\mathrm{d}\zeta}{\mu(\zeta)} \right]. \tag{3.9}$$

Observe that each laminar flow (3.14) can be thought of as a linear combination of Couette and Poiseuille flows. Finally we normalize (3.8) and (3.9) by dividing them by V, where, in the former case, V is the velocity of the upper plate, and, in the latter,

$$V = A_0 \left[\int_{-1/2}^{\bar{z}} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta - \frac{\displaystyle\int_{-1/2}^{1/2} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta}{\displaystyle\int_{-1/2}^{1/2} \frac{\mathrm{d}\zeta}{\mu(\zeta)}} \int_{-1/2}^{\bar{z}} \frac{\mathrm{d}\zeta}{\mu(\zeta)} \right] \tag{3.10}$$

is the velocity at the stationary surface

$$z = \bar{z} = \left[\int_{-1/2}^{1/2} \frac{\zeta}{\mu(\zeta)} \mathrm{d}\zeta \right] \left[\int_{-1/2}^{1/2} \frac{\mathrm{d}\zeta}{\mu(\zeta)} \right]^{-1}. \tag{3.11}$$

3.3 Laminar flows in polymer melts

We now consider the exponential dependence of viscosity on temperature and pressure proposed by Laun for polymer melts [36],

$$\mu = \mu_0 \exp[\beta(p - p_0) - \gamma(T - T_0)], \tag{3.12}$$

where the non-negative numbers β and γ are the pressure and temperature coefficients of viscosity. Obviously, for $\beta = 0$ and $\gamma = 0$ (3.12) yields the classical case with constant viscosity. According to (3.3) and $(3.7)_1$, the dimensionless viscosity (3.12) is given by

$$\mu = \exp(\Gamma z) \tag{3.13}$$

with $\Gamma = \gamma(T_1 - T_2) - \beta\rho_0 gd$, and hence $(3.7)_3$ becomes

$$v = -\left[\frac{A_0}{\Gamma^2}(\Gamma z + 1) + \frac{k_1}{\Gamma}\right]\exp(-\Gamma z) + k_2, \tag{3.14}$$

where

$$k_1 = \frac{(V_2 - V_1)\Gamma}{2\sinh(\Gamma/2)} + \frac{A_0}{2}\coth(\Gamma/2) - \frac{A_0}{\Gamma}$$

and

$$k_2 = \frac{V_2\exp(\Gamma/2) - V_1\exp(-\Gamma/2)}{2\sinh(\Gamma/2)} + \frac{A_0}{2\Gamma\sinh(\Gamma/2)}.$$

The Couette and Poiseuille flows are, respectively, given by

$$v = \frac{V}{\sinh(\Gamma/2)}\left[\cosh(\Gamma/2) - \exp(-\Gamma z)\right]; \tag{3.15}$$

and

$$v = \frac{A_0}{2\Gamma\sinh(\Gamma/2)}\left\{1 - \left[2z\sinh(\Gamma/2) + \cosh(\Gamma/2)\right]\exp(-\Gamma z)\right\}. \tag{3.16}$$

We now remark that in the limit as $\Gamma \to 0$ (3.15) and (3.16) give the Couette and the Poiseuille flows in a fluid whose viscosity is assumed to be constant (see for example [17] page 154):

$$v = 2Vz \quad \text{(Couette flow)}$$

and

$$v = \frac{A_0}{2}\left(z^2 - \frac{1}{4}\right) \quad \text{(Poiseuille flow)}.$$

Next we normalize (3.15) and (3.16) by dividing them by V, where, in the former case, V is the velocity at the top, and, in the latter,

$$V = \frac{A_0}{2\Gamma\sinh(\Gamma/2)}\left\{1 - \frac{2}{\Gamma}\sinh(\Gamma/2)\exp\left[\frac{\Gamma\cosh(\Gamma/2)}{2\sinh(\Gamma/2)} - 1\right]\right\}$$

is the velocity at the stationary surface

$$z = \frac{1}{\Gamma} - \frac{\cosh(\Gamma/2)}{2\sinh(\Gamma/2)}.$$

Normalized velocity profiles of Couette and Poiseuille flows are plotted for different values of the non-dimensional parameter Γ in Figures 3.1-3.5. We observe that the normalized velocity profiles of Couette flow are convex for negative values of Γ, that is when the dependence of viscosity on pressure is stronger than that on temperature. Moreover, for such values of Γ,

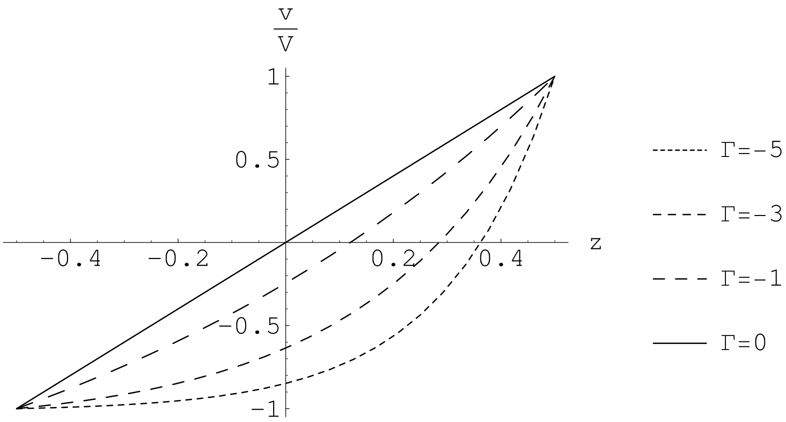

Figure 3.1: Normalized velocity profiles of Couette flow for different non-positive values of the parameter Γ.

viscosity is a decreasing function of the height z so that the fluid layer having velocity oriented as the velocity of the upper plate (i.e. as $\mathbf{i}$) is thinner than that with velocity oriented as the velocity of the lower one (i.e. as $-\mathbf{i}$). On the contrary, for positive values of Γ, that is when the dependence of viscosity on temperature is stronger than that on pressure, the normalized velocity profiles of Couette flow are concave and the fluid layer with velocity oriented as $\mathbf{i}$ is thicker than that having velocity oriented as $-\mathbf{i}$. In Figure 3.3 we show how the thickness d_+ of the fluid layer with velocity oriented as the velocity of the upper plate depends on the parameter Γ.

In Poiseuille flow, instead, for negative values of Γ the velocity profiles attain their maximum at $z_{\max} \in]0, 1/2[$, and as Γ decreases $z_{\max}$ approaches $z = 1/2$ where both pressure due to gravity and viscosity are minimum (see $(3.7)_1$ and (3.13)). For positive values of Γ the velocity profiles attain their maximum at $z_{\max} \in] - 1/2, 0[$, and, as shown in Figure 3.6, as Γ increases $z_{\max}$ approaches $z = -1/2$ at which temperature is maximum whereas viscosity is minimum.

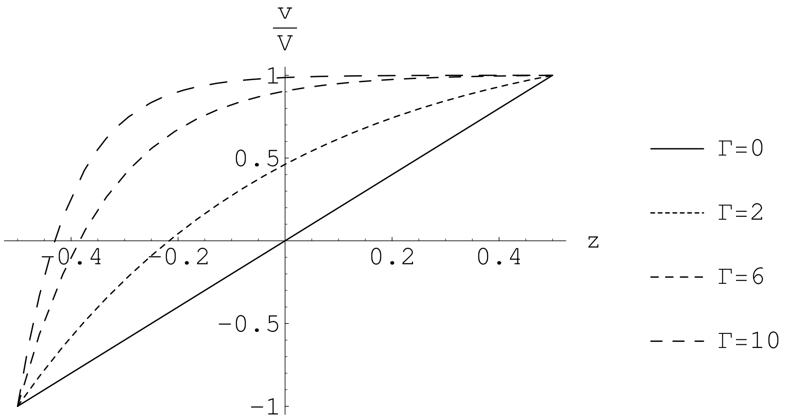

Figure 3.2: Normalized velocity profiles of Couette flow for different non-negative values of the parameter Γ.

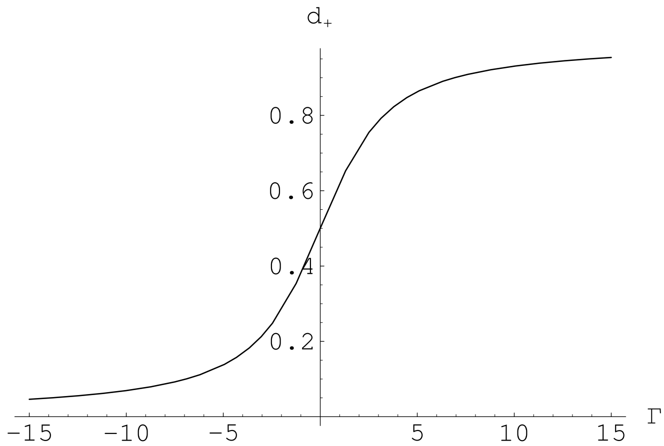

Figure 3.3: Thickness d_+ as function of Γ. For negative values of Γ, $d_+ < 1/2$, decreases as Γ decreases and in the limit as $\Gamma \to -\infty$ tends to zero. If $\Gamma = 0$, in particular in the classical case $\beta = 0$ and $\gamma = 0$, $d_+ = 1/2$. For positive value of Γ, $d_+ > 1/2$, increases as Γ increases and in the limit as $\Gamma \to +\infty$ tends to 1.

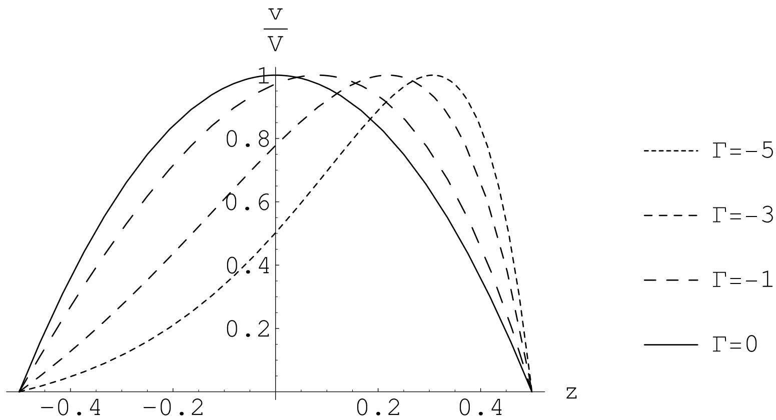

Figure 3.4: Normalized velocity profiles of Poiseuille flow for different non-positive values of the parameter Γ.

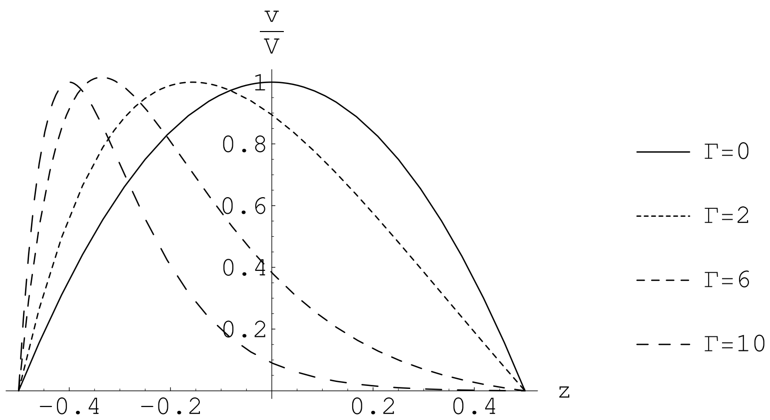

Figure 3.5: Normalized velocity profiles of Poiseuille flow for different non-negative values of the parameter Γ.

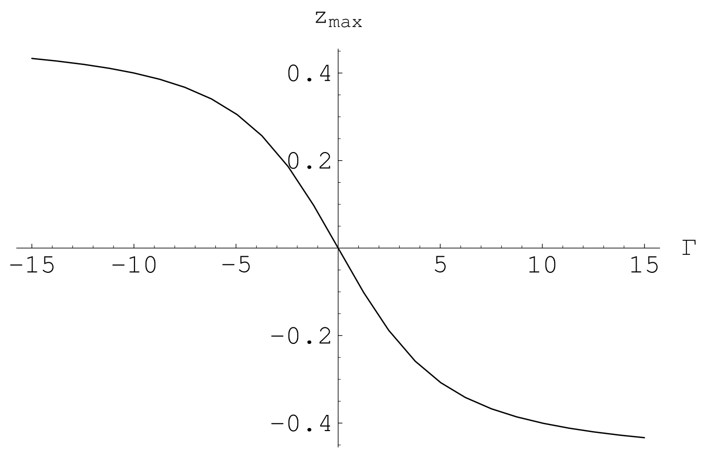

Figure 3.6: The point $z_{\max}$ as function of Γ. For negative values of Γ, $z_{\max} \in]0, 1/2[$, increases as Γ decreases and in the limit as $\Gamma \to -\infty$ tends to $1/2$. If $\Gamma = 0$, in particular in the classical case $\beta = 0$ and $\gamma = 0$, $z_{\max} = 0$. For positive values of Γ, $z_{\max} \in] - 1/2, 0[$, decreases as Γ increases and in the limit as $\Gamma \to +\infty$ tends to $-1/2$.

3.4 Couette and Poiseuille flows of bitumen

Bitumen is a hydrocarbon mixture usually produced by vacuum distillation of petroleum crude oils. The chemical composition of bitumen is very complex and thus bitumen can be separated into four fractions: saturates, aromatics, resins and alphaltenes [47]. If the proportion of these fractions vary, the resulting physical properties and microstructure of bitumen may be quite different.

Asphalt is a composite mixture of bitumen with mineral aggregates, widely used for road paving applications. The mechanical properties of asphalt are related to the rheological characteristics of bitumen, because it forms the continuous matrix and is the only deformable component. In addition, the workability (easiness of mixing, laying and compacting operations) of hot rolled asphalt depends on bitumen viscosity, among other factors [96]. Thus, bitumen is a Newtonian fluid when handled and mixed with mineral aggregates at high temperatures.

Compaction is probably the most crucial stage in the construction of road pavements because improving compaction can result in a significant improvement in road resistance to cracking and deformation. Asphalt compaction is a consequence of the static pressure that the deadweight of the roller exerts on the road surface. It is apparent that the performance of asphalt compaction will depend on bitumen viscosity. Both temperature and pressure exert an important influence on bitumen viscosity and, consequently, on its workability and road performance. The FMT model, proposed by Tschoegl, Knauss and Emri [91], describes the evolution of bitumen Newtonian viscosity, in the range between 60 °C and 160 °C and at any given differential pressure in the range 0-400 bars, fairly well. The FMT model is given as:

$$\log\left(\frac{\mu}{\mu_0}\right) = -\frac{c_1^{00}[T - T_0 - \theta(p)]}{c_2(p) + [T - T_0 - \theta(p)]}, \tag{3.17}$$

being

$$\theta(p) = c_3(p)\ln\left(\frac{1 + c_4 p}{1 + c_4 p_0}\right) - c_5(p)\ln\left(\frac{1 + c_6 p}{1 + c_6 p_0}\right), \tag{3.18}$$

$$c_1^{00} = \frac{B}{2.303 f_0}, \tag{3.19}$$

$$c_2(p) = \frac{f_0}{\alpha_f(p)}, \tag{3.20}$$

$$c_3(p) = \frac{1}{k_e \alpha_f(p)}, \tag{3.21}$$

$$c_4 = \frac{k_e}{K_e^*}, \tag{3.22}$$

$$c_5(p) = \frac{1}{k_\phi \alpha_f(p)}, \tag{3.23}$$

$$c_6 = \frac{k_\phi}{K_\phi^*}, \tag{3.24}$$

$$\alpha_f = \alpha_f^*\left(1 - \frac{mp}{K_e^* + k_e p}\right) - m\alpha_\phi^* p\left(\frac{1}{K_e^* + k_e p} - \frac{1}{K_\phi^* + k_\phi p}\right), \tag{3.25}$$

where, μ_0 is the viscosity at the reference temperature and atmospheric pressure; f_0 is the fractional free-volume at the reference temperature; B is a constant that normally is taken to be 1; $\alpha_f(p)$ is the expansivity of the free-volume, considered pressure dependent and temperature independent; α_f^* is the expansivity of the free-volume at zero differential pressure and temperature of reference, α_ϕ^* is the expansivity of the occupied volume at zero differential pressure and temperature of reference; K_e^* and K_ϕ^* are the bulk moduli of the entire and occupied volume at zero differential pressure and temperature of reference; k_e, k_ϕ and m are proportionality constants, which are indipendent of temperature and pressure; the superscript 00 indicates temperature and pressure of reference. The values of all the FMT model parameters for bitumen are shown in Table 3.1 (see also [46, 47]).

Then by non-dimensionalizing (3.17)-(3.25) by means of (3.3) and by inserting the resulting dimensionless viscosity into (3.8)-(3.11) we can plot

ρ_0	$991\ \mathrm{kg}\cdot\mathrm{m}^{-3}$	$(T_1 - T_2)/d$	$3\cdot 10^{-2}\ \mathrm{K{\cdot}m}^{-1}$
μ_0	228.3 Pa·s	B	1
f_0	0.069	k_e	3.256
K_e^*	$1.531\cdot 10^4$ bar	k_ϕ	0.322
K_ϕ^*	$2.279\cdot 10^4$ bar	α_f^*	$6.335\cdot 10^{-4}\ \mathrm{K}^{-1}$
α_ϕ^*	$9.631\cdot 10^{-4}\ \mathrm{K}^{-1}$	m	3.508

Table 3.1: Values of the different parameters of the FMT model for bitumen (60/70 penetration grade) at the reference temperature $T_0 = 60$ °C and the reference pressure $p_0 = 1$ bar.

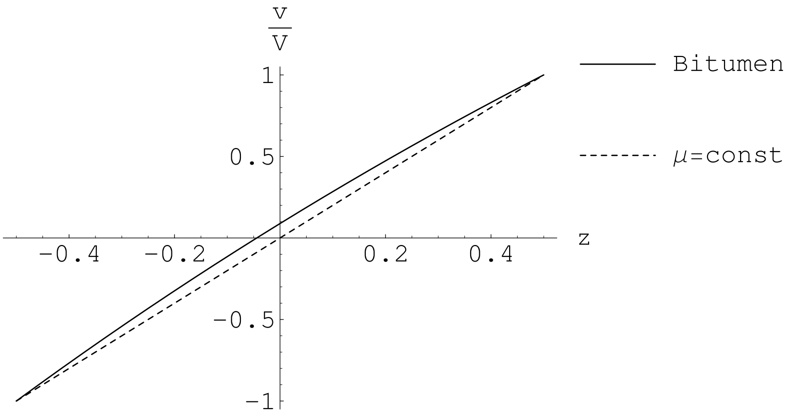

Figure 3.7: Normalized velocity profile of Couette flow in bitumen compared with the Couette flow in a fluid with constant viscosity.

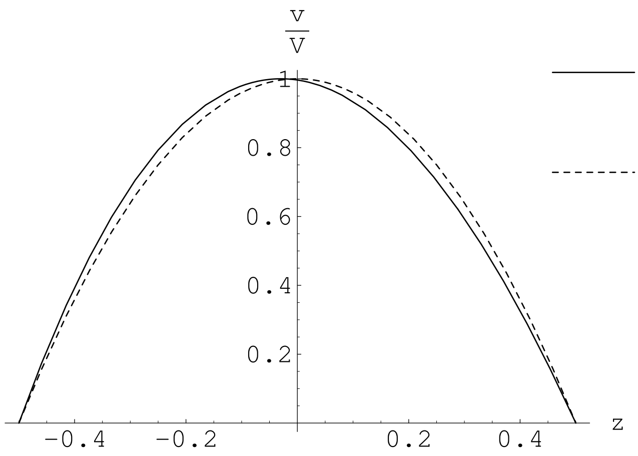

Figure 3.8: Normalized velocity profile of Poiseuille flow in bitumen compared with the Poiseuille flow in a fluid with constant viscosity.

the nomalized velocity profiles as shown in Fugures 3.7 and 3.8, respectively. These velocity profiles differ not so much from the classical case ($\mu = \mathrm{const}$) in spite of the intricate model for bitumen viscosity given by (3.17)-(3.25). Moreover, since the normalized velocity profile in Couette flow is concave, and since the normalized velocity profile in Poiseuille flow attains its maximum approximately at $z = -0.0295$, we may conclude that the dependence of bitumen viscosity on temperature is stronger than that on pressure (see the discussion at the end of section 3.3).

Remark 3.1. In this chapter the approximate equations derived in section 2.4 are used to find the laminar flows in polymer melts and in bitumen. We think that these equations will have relevance to geophysical flows (wherein the viscosity changes with the depth of the fluid) as the approximation established in section 2.4 is valid when the dimensionless quantity $\alpha(T_1 - T_2)$ is small and does not need the fluid layer being sufficiently thin.

Chapter 4

Stability analysis of the Rayleigh-Bénard convection for a fluid with temperature and pressure dependent viscosity

4.1 Introduction

Consider a horizontal layer of fluid in which an adverse temperature gradient is mantained by heating the underside. The temperature gradient thus mantained is qualified as adverse since, on account of thermal expansion, the fluid at the bottom will be lighter than the fluid at the top. The basic state is then one of rest with light fluid below heavy fluid. When the adverse temperature gradient is great enough, the stabilizing effects of viscosity and thermal conductivity are overcome by the destabilizing buoyancy, and an overturning instability ensues as thermal convection. Convective instability was first described by Thomson in 1882 but the first experiments were made by Bénard in 1900. The experiments of Bénard established that the motions which ensue on surpassing the critical temperature gradient have a cellular stationary character. At the onset of instability the fluid layer resolves itself into a number of cells; and if the experiment is performed with sufficient care, the cells become equal and they align themselves to form a regular hexagonal pattern. This is called *Bénard convection* although Pearson [61] proved that most of the motions observed by Bénard were driven by the variation of surface tension with temperature and not by thermal instability of light fluid below heavy fluid.

The theoretical foundations for a correct interpretation of the convective instability were laid by Lord Rayleigh [69] who chose equations of motions and boundary conditions to model the experiments of Bénard and derived the linear equations for normal modes. He then showed that instability would occur only when the adverse temperature gradient was so large that the dimensionless parameter

$$\mathscr{R} = \frac{g\alpha d^4}{\kappa\nu}\left|\frac{\mathrm{d}T}{\mathrm{d}z}\right|,$$

now called Rayleigh number, exceeded a certain critical value. Here g is the acceleration due to gravity, α the coefficient of volumetric thermal expansion of the fluid, d the depth of the fluid layer, κ its thermal diffusivity, ν its kinematic viscosity and $|\mathrm{d}T/\mathrm{d}z|$ the magnitude of the vertical temperature gradient. Further theoretical and numerical studies of thermal convection for fluids with constant viscosity can be found in [11, 17] and references therein.

Fundamental early paper on convection in temperature-dependent viscosity fluids is that of Palm and coworkers [60] in which the following linear relationship

$$\nu(T) = \nu_0[1 - \gamma(T - T_0)],$$

ν_0, γ and T_0 being positive constants, is adopted. Richardson and Straughan [72] developed a conditional nonlinear stability analysis for such fluids and the result they obtained is very sharp in that it derives coincidence of the nonlinear stability and linear instability Rayleigh number thresholds. Capone and Gentile [8, 10] also develop a nonlinear stability analysis for fluids whose temperature-dependent kinematic viscosity is of the form

$$\nu(T) = \nu_0 \exp[-\gamma(T - T_0)],$$

whereas in [9] they treat a very general viscosity of the type

$$\nu(T) = \nu_0 f(T),$$

in which f is a convex non-increasing function. See also [15, 86, 92] and references therein for other important studies on the thermal convection for fluids with temperature-dependent viscosity.

On the contrary the stability analysis of the Bénard problem for fluids with pressure-dependent viscosity, to our knowledge, has not been received the same attention although it could be of practical interest in geophysics and in polymer melt processing. When the dependence of viscosity on pressure is taken into account, the Oberbeck-Boussinesq equations, i.e. the approximate equations of motion of a heat-conducting viscous fluid under the action of gravity, must be slightly modified as one needs to distinguish between the pressure due to gravity and the pressure due to the thermal expansion of the fluid, only the former contributes to variations in viscosity at a first approximation as we have shown in section 2.4. Then, by using the Oberbeck-Boussinesq-type equations we have derived in section 2.4 under the assumption that the coefficient of volumetric thermal expansion α, the heat conductivity k and the specific heat at constant pressure c_p are constants (such an assumption is reasonable as we have seen in section 3.1), we study the stability of the conduction solution in fluids whose viscosity is an analytic function of both temperature and pressure. In particular we first introduce the dimensionless perturbation equations of the Bénard problem for such a class of fluids. Thus we prove that the principle of exchange of stabilities holds and hence instability sets in as stationary convection. Furthermore, by following a standard procedure, we show how to find the critical Rayleigh number, the linear stability-instability threshold, by appealing to a variational analysis. Finally we study the nonlinear stability of the basic conduction solution by employing the energy method, and prove that the thresholds for linear theory and energy analysis coincide, provided the initial disturbance to the temperature field meets a specific restriction. We end this chapter with numerical results when the viscosity depends on temperature and pressure as in (3.12).

4.2 The problem

Let $Oxyz$ be a Cartesian frame of reference with fundamental unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively, $\mathbf{k}$ pointed vertically upward in a direction opposed to that in which gravity acts. Let $\Omega_d = \mathbb{R}^2 \times (0, d)$ $(d > 0)$ be a horizontal layer of fluid whose viscosity is an analytic function of pressure and temperature and assume that the top and bottom surfaces of the fluid are held at constant temperature T_2 and T_1 $(T_1 > T_2)$, respectively. The equations governing the fluid motion in Ω_d are:

$$\begin{cases} \nabla p + \rho_0 g \mathbf{k} = 0 \\ \rho_0 \mathbf{v}_t + \rho_0 \mathbf{v} \cdot \nabla \mathbf{v} = -\alpha(T_1 - T_2)\nabla P + \mu(p, T)\Delta \mathbf{v} \\ \qquad\qquad +2\mathbf{D} \cdot \nabla \mu(p, T) + \rho_0 g \alpha (T - T_2)\mathbf{k} \\ \operatorname{div} \mathbf{v} = 0 \\ T_t + \mathbf{v} \cdot \nabla T = \kappa \Delta T \end{cases} \tag{4.1}$$

where ρ_0 is the density at the reference temperature T_2, $\kappa = k/(\rho_0 c_p)$ is the thermal diffusivity, g and p are, respectively, the acceleration and the pressure field due to gravity, P is the pressure due to the thermal expansion of the fluid and T is the temperature. Equations (4.1) have been established in section 2.4. The appropriate boundary conditions to append to system (4.1) are

$$\begin{cases} T(x_1, x_2, 0, t) = T_1, \quad T(x_1, x_2, d, t) = T_2, \\ p(x_1, x_2, d, t) = p_0 \end{cases} \tag{4.2}$$

where p_0 is the reference pressure. Our aim is the study of stability of the steady static conduction solution m_0 to (4.1)-(4.2):

$$\begin{cases} \bar{p} = -\rho_0 g(z-d) + p_0 \\ \bar{\mathbf{v}} = \mathbf{0} \\ \bar{T} = -\dfrac{T_1 - T_2}{d} z + T_1 \\ \bar{P} = -\rho_0 g z \left(\dfrac{z}{2d} - 1\right) + P_0. \end{cases} \tag{4.3}$$

In order to study the stability of the conduction solution m_0 we introduce the perturbations $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, θ, p_1 and P_1 to $\bar{\mathbf{v}}$, $\bar{T}$, $\bar{p}$ and $\bar{P}$, respectively, i.e.,

$$\mathbf{v} = \bar{\mathbf{v}} + \mathbf{u}, \quad T = \bar{T} + \theta, \quad p = \bar{p} + p_1, \quad P = \bar{P} + P_1.$$

Setting $\mathbf{d} = [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]/2$, from (3.1) the perturbations are found to satisfy

$$\begin{cases} \nabla p_1 = 0 \\ \rho_0 \mathbf{u}_t + \rho_0 \mathbf{u} \cdot \nabla \mathbf{u} = -\alpha(T_1 - T_2)\nabla P_1 + \mu(\bar{p} + p_1, \bar{T} + \theta)\Delta\mathbf{u} \\ \qquad\qquad +2\mathbf{d} \cdot \nabla\mu(\bar{p} + p_1, \bar{T} + \theta) + \rho_0 g \alpha \theta \mathbf{k} \\ \operatorname{div} \mathbf{u} = 0 \\ \theta_t + \mathbf{u} \cdot \nabla\theta - \dfrac{T_1 - T_2}{d} w = \kappa \Delta\theta \end{cases} \tag{4.4}$$

in $\mathbb{R}^2 \times (0, d) \times (0, +\infty)$. To the previous system we append the initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, \quad \theta(\mathbf{x}, 0) = \theta_0(\mathbf{x}), \tag{4.5}$$

and the boundary conditions

$$\theta(x, y, 0, t) = \theta(x, y, d, t) = 0, \quad p_1(x, y, 0, t) = 0 \tag{4.6}$$

as the surface $z = 0$ is mantained at constant temperature whereas the surface $z = d$ is mantained at constant pressure as well as at constant temperature. In (4.5) $\mathbf{u}_0$ and θ_0 are regular fields, $\mathbf{u}_0$ being divergence-free. From $(4.4)_1$ and $(4.6)_2$ it readily follows that $p_1 \equiv 0$. As concerns the boundary conditions for the perturbation to velocity $\mathbf{u}$ we shall distinguish two kinds of bounding surfaces: *rigid surfaces* on which no slip occurs and *free surfaces* on which no tangential stresses act (see [11] for details).

For rigid bounding surfaces

$$\mathbf{u} = \mathbf{0} \quad \text{at } z = 0, d. \tag{4.7}$$

Since this condition must be satisfied for all x and y on the rigid surfaces $z = 0, d$, from the equation of continuity $(4.4)_3$ it follows that

$$\frac{\partial w}{\partial z} = 0 \quad \text{at } z = 0, d.$$

For free bounding surfaces

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{at } z = 0, d, \tag{4.8}$$

by which, differentiating the equation of continuity $(4.4)_3$ with respect to z, we deduce that

$$\frac{\partial^2 w}{\partial z^2} = 0 \quad \text{at } z = 0, d.$$

Returning to equations (4.4), we non-dimensionalize them by introducing the following dimensionless quantities:

$$\left.\begin{aligned} \mathbf{x}^* = \frac{\mathbf{x}}{d}, \quad t^* = \frac{\mu_0}{\rho_0 d^2} t, \quad \mathbf{u}^* = \frac{\rho_0 d}{\mu_0}\mathbf{u}, \quad \mu^* = \frac{\mu}{\mu_0}, \\ \bar{p}^* = \frac{\bar{p} - p_0}{\rho_0 g d} = -(z^* - 1), \quad \bar{T}^* = \frac{\bar{T} - T_2}{T_1 - T_2} = -(z^* - 1), \\ P_1^* = \frac{\alpha(T_1 - T_2)\rho_0 d^2}{\mu_0^2} P_1, \quad \theta^* = \frac{\rho_0 d}{\mu_0}\sqrt{\frac{\alpha \rho_0 g d \kappa}{\mu_0 (T_1 - T_2)}}\,\theta, \\ \mathscr{R} = R^2 = \frac{\alpha(T_1 - T_2)\rho_0 g d^3}{\mu_0 \kappa}, \quad Pr = \frac{\mu_0}{\rho_0 \kappa}, \end{aligned}\right\} \tag{4.9}$$

where $\mu_0 = \mu(p_0, T_2)$ is the viscosity at the reference state (p_0, T_2). With this scaling the non-dimensional form of (4.4) becomes (omitting all asterisks)

$$\left\{\begin{aligned} & \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P_1 + \mu(\bar{p}, \bar{T} + \theta)\Delta \mathbf{u} \\ & \qquad\qquad + 2\mathbf{d} \cdot \nabla \mu(\bar{p}, \bar{T} + \theta) + R\theta \mathbf{k} \\ & \operatorname{div} \mathbf{u} = 0 \\ & Pr(\theta_t + \mathbf{u} \cdot \nabla \theta) - Rw = \Delta \theta \end{aligned}\right. \tag{4.10}$$

in $\mathbb{R}^2 \times (0,1) \times (0, +\infty)$ with boundary conditions

$$\theta = 0 \quad \text{at } z = 0, 1,$$

and

$$\mathbf{u} = \mathbf{0} \quad \text{at } z = 0, 1,$$

for rigid boundaries, or

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0 \quad \text{at } z = 0, 1,$$

for free bounding surfaces. In (4.10)

$$\mathscr{R} = R^2 = \frac{\alpha(T_1 - T_2)\rho_0 g d^3}{\mu_0 \kappa}$$

is the Rayleigh number and

$$Pr = \frac{\mu_0}{\rho_0 \kappa}$$

is the Prandtl number. Note that the Rayleigh number is positive since the lower boundary is hotter than the upper one and is seen to be the characteristic ratio of the buoyancy to the viscous forces. Also note that the Prandtl number is an intrinsic property of the fluid; it measures the ratio of the molecular diffusion of momentum and heat.

From now on, as usual, we shall assume that the perturbations $\mathbf{u}$, θ and P_1 have periods $2\pi/a_x$ and $2\pi/a_y$ in the x and y directions ($a_x > 0$, $a_y > 0$), denote by Ω the period cell

$$\Omega = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [0, 1]$$

and by $a = (a_x^2 + a_y^2)^{1/2}$ the two-dimensional wave number. Moreover, since the stability of m_0 makes sense only in a class of solutions of (4.10) in which the zero solution $u = v = w = \theta = P_1 = 0$ is unique, for free bounding surfaces we exclude any other solution by requiring the usual 'average velocity conditions' (see [33])

$$\int_\Omega u \mathrm{d}\Omega = \int_\Omega v \mathrm{d}\Omega = 0. \tag{4.11}$$

4.3 Linear stability analysis

Since we have assumed that the viscosity is an analytic function of the temperature and pressure, for sufficiently small disturbances we can expand μ in the following manner:

$$\mu(\bar{p}, \bar{T} + \theta)\Delta\mathbf{u} = \left[\sum_{n=0}^{+\infty} \frac{1}{n!} \frac{\partial^n \mu}{\partial T^n}(\bar{p}, \bar{T})\theta^n\right] \Delta\mathbf{u} \approx \mu(z)\Delta\mathbf{u}$$

and

$$
\begin{aligned}
2\mathbf{d}\cdot\nabla\mu(\bar{p},\bar{T}+\theta) &= 2\mathbf{d}\cdot\left\{\sum_{n=0}^{+\infty}\frac{1}{n!}\nabla\left[\frac{\partial^n\mu}{\partial T^n}(\bar{p},\bar{T})\theta^n\right]\right\}\\
&\approx \mu'(z)\left[\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\mathbf{i}+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\mathbf{j}+2\frac{\partial w}{\partial z}\mathbf{k}\right],
\end{aligned}
$$

where

$$\mu(z)=\mu(\bar{p},\bar{T}) \tag{4.12}$$

and the prime denotes differentiation with respect to z. Thus linearizing (4.10) we obtain

$$
\begin{cases}
\mathbf{u}_t = -\nabla P_1+\mu(z)\Delta\mathbf{u}+\mu'(z)\Big[\Big(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\Big)\mathbf{i}\\
\qquad +\Big(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\Big)\mathbf{j}+2\frac{\partial w}{\partial z}\mathbf{k}\Big]+R\theta\mathbf{k}\\
\operatorname{div}\mathbf{u}=0\\
Pr\theta_t-Rw=\Delta\theta.
\end{cases} \tag{4.13}
$$

We can easily eliminate the pressure P_1 and the dependent variables u and v. The curl of equation $(4.13)_1$ gives

$$\frac{\partial\boldsymbol{\omega}}{\partial t}=\mu(z)\Delta\boldsymbol{\omega}+\mu'(z)\frac{\partial\boldsymbol{\omega}}{\partial z}+\big[R\nabla\theta-\mu'(z)\Delta\mathbf{u}-2\mu''(z)\mathbf{d}\cdot\mathbf{k}\big]\times\mathbf{k} \tag{4.14}$$

where the vorticity $\boldsymbol{\omega}=\nabla\times\mathbf{u}$. The curl of equation (4.14) in turn gives, after use of equation $(4.13)_2$,

$$
\begin{aligned}
\frac{\partial}{\partial t}\Delta\mathbf{u} &= 2\mu'(z)\Delta\frac{\partial\mathbf{u}}{\partial z}+\mu(z)\Delta\Delta\mathbf{u}+\mu''(z)\frac{\partial^2\mathbf{u}}{\partial z^2}-\mu''(z)\Delta w\mathbf{k}+\mu''(z)\nabla\frac{\partial w}{\partial z}\\
&\quad +R\Delta\theta\mathbf{k}-R\nabla\frac{\partial\theta}{\partial z}+\mu'(z)\Delta\boldsymbol{\omega}\times\mathbf{k}+\mu''(z)(\Delta\mathbf{u}-\Delta w\mathbf{k})\\
&\quad +2\mu'''(z)\left(\mathbf{d}\cdot\mathbf{k}-\frac{\partial w}{\partial z}\mathbf{k}\right)+\mu''(z)\frac{\partial\boldsymbol{\omega}}{\partial z}\times\mathbf{k}.
\end{aligned}
$$

In particular

$$\frac{\partial}{\partial t}\Delta w=2\mu'(z)\Delta\frac{\partial w}{\partial z}+\mu(z)\Delta\Delta w+\mu''(z)\frac{\partial^2 w}{\partial z^2}-\mu''(z)\Delta_1 w+R\Delta_1\theta$$

where by

$$\Delta_1=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

we denote the horizontal Laplacian. From the equation of continuity $(4.13)_2$ one can readily deduce that

$$\Delta_1 u = -\frac{\partial^2 w}{\partial x \partial z} - \frac{\partial \zeta}{\partial y} \tag{4.15}$$

and

$$\Delta_1 v = -\frac{\partial^2 w}{\partial y \partial z} + \frac{\partial \zeta}{\partial x}, \tag{4.16}$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

is the vertical component of vorticity. This is given by the vertical component of equation (4.14), namely

$$\frac{\partial \zeta}{\partial t} = \mu(z)\Delta\zeta + \mu'(z)\frac{\partial \zeta}{\partial z}. \tag{4.17}$$

From (4.7) and (4.8) the boundary conditions for ζ are

$$\zeta = 0 \quad \text{at } z = 0, 1, \quad \text{for rigid surfaces,}$$

$$\frac{\partial \zeta}{\partial z} = 0 \quad \text{at } z = 0, 1, \quad \text{for free surfaces.}$$

So u and v can be found by solving the Poisson equations (4.15), (4.16) when w has been found by solving the system

$$\begin{cases} \dfrac{\partial}{\partial t}\Delta w = 2\mu'(z)\Delta\dfrac{\partial w}{\partial z} + \mu(z)\Delta\Delta w + \mu''(z)\dfrac{\partial^2 w}{\partial z^2} \\ \qquad -\mu''(z)\Delta_1 w + R\Delta_1\theta \\ Pr\theta_t - Rw = \Delta\theta \end{cases} \tag{4.18}$$

and ζ by solving the diffusion equation (4.17).

Since the coefficients in equations (4.18) depend only on z, the equations admit solutions which depend on x, y and t exponentially. We consider therefore solutions of the form:

$$\begin{cases} w(x,y,z,t) = W(z)\exp[\mathrm{i}(a_x x + a_y y) + ct] \\ \theta(x,y,z,t) = \Theta(z)\exp[\mathrm{i}(a_x x + a_y y) + ct], \end{cases} \tag{4.19}$$

in which it is understood that the real parts of these expressions must be taken to obtain physical quantities. The wave speed c may be complex, i.e. $c = c_r + \mathrm{i}c_i$, and the expressions (4.19) thus represent waves which travel in the direction $(a_x, a_y, 0)$ with phase speed $c_\mathrm{i}/(a_x^2 + a_y^2)^{1/2}$ and which grow or decay in time like $\exp(c_r t)$. Such a wave is stable if $c_r \leq 0$, unstable if $c_r > 0$, and neutrally stable if $c_r = 0$.

If we now let $\mathrm{D} = \mathrm{d}/\mathrm{d}z$ and $a = (a_x^2 + a_y^2)^{1/2}$, then on substituting the expressions (4.19) into equations (4.18) we obtain the system of ordinary differential equations

$$\begin{cases} c(\mathrm{D}^2 - a^2)W = 2\mu'(z)\mathrm{D}(\mathrm{D}^2 - a^2)W + \mu(z)(\mathrm{D}^2 - a^2)^2 W \\ \qquad\qquad +\mu''(z)(\mathrm{D}^2 + a^2)W - Ra^2\Theta \\ cPr\Theta - RW = (\mathrm{D}^2 - a^2)\Theta, \end{cases} \tag{4.20}$$

to which we add the boundary conditions

$$W = \mathrm{D}W = \Theta = 0 \quad \text{at } z = 0, 1, \quad \text{for rigid surfaces,} \tag{4.21}$$

or

$$W = \mathrm{D}^2 W = \Theta = 0 \quad \text{at } z = 0, 1, \quad \text{for free surfaces.} \tag{4.22}$$

Denoting by the superscript * the complex conjugate, multiplying $(4.20)_1$ by W^*, $(4.20)_2$ by $a^2\Theta^*$, summing and integrating over the interval $[0, 1]$, we have

$$\begin{aligned} c\int_0^1 [|\mathrm{D}W|^2 + a^2(|W|^2 + Pr|\Theta|^2)]\mathrm{d}z &= a^2 R\int_0^1 (\Theta W^* + W\Theta^*)\mathrm{d}z \\ &- \int_0^1 \mu(z)\left(|(\mathrm{D}^2 + a^2)W|^2 + 4a^2|\mathrm{D}W|^2\right)\mathrm{d}z \\ &- a^2\int_0^1 (|\mathrm{D}\Theta|^2 + a^2|\Theta|^2)\mathrm{d}z. \end{aligned} \tag{4.23}$$

The right hand side of (4.23) is real and then taking the imaginary part of (4.23) we find

$$c_i = 0.$$

Therefore the linearized equations for Bénard convection satisfy the principle of exchange of stabilities even when the fluid viscosity depends analytically on temperature and pressure. Thus, to find the instability boundary, the lowest value of $\mathscr{R} = R^2$ for which $c > 0$, we solve (4.20) for the smallest eigenvalue $R_L(a)$ with $c > 0$ (see [87]), that is we find the least eigenvalue $R_L(a)$ of the characteristic-value problem which gives the neutrally stable states

$$\begin{cases} 2\mu'(z)\mathrm{D}(\mathrm{D}^2 - a^2)W + \mu(z)(\mathrm{D}^2 - a^2)^2 W \\ \qquad\qquad +\mu''(z)(\mathrm{D}^2 + a^2)W = Ra^2\Theta \\ (\mathrm{D}^2 - a^2)\Theta + RW = 0 \end{cases} \tag{4.24}$$

with boundary conditions (4.21) or (4.22).

We now prove that for marginal stable disturbances

$$\frac{1}{R_L(a)} = \max_{(W,\Theta)\in\mathcal{H}} \frac{\mathcal{I}(W,\Theta)}{\mathcal{D}(W,\Theta)}, \tag{4.25}$$

$$\mathcal{I}(W,\Theta) = a^2 \int_0^1 (W\Theta^* + \Theta W^*)\mathrm{d}z, \tag{4.26}$$

$$\begin{aligned}\mathcal{D}(W,\Theta) = &\int_0^1 \mu(z)\left(|(\mathrm{D}^2+a^2)W|^2 + 4a^2|\mathrm{D}W|^2\right)\mathrm{d}z \\ &+ a^2\left(\int_0^1 |\mathrm{D}\Theta|^2\mathrm{d}z + a^2\int_0^1 |\Theta|^2\mathrm{d}z\right)\end{aligned} \tag{4.27}$$

and $\mathcal{H}$ denotes the set of the kinematically admissible disturbances:

$$\mathcal{H} = \left\{(W,\Theta) \in \mathrm{H}^2(0,1)\times\mathrm{H}^1(0,1) : W = \mathrm{D}W = \Theta = 0 \ \text{at}\ z = 0,1\right\}$$

for rigid boundaries, or

$$\mathcal{H} = \left\{(W,\Theta) \in \mathrm{H}^2(0,1)\times\mathrm{H}^1(0,1) : W = \mathrm{D}^2W = \Theta = 0 \ \text{at}\ z = 0,1\right\}$$

for free surfaces.

By (4.26) and (4.27), (4.23) becomes

$$c\int_0^1 [|\mathrm{D}W|^2 + a^2(|W|^2 + Pr|\Theta|^2)]\mathrm{d}z = \left[R\frac{\mathcal{I}(W,\Theta)}{\mathcal{D}(W,\Theta)} - 1\right]\mathcal{D}(W,\Theta) \tag{4.28}$$

by which we readily deduce that if

$$R \le \left[\max_{(W,\Theta)\in\mathcal{H}} \frac{\mathcal{I}(W,\Theta)}{\mathcal{D}(W,\Theta)}\right]^{-1},$$

then the modes of two-dimensional wave number a are linearly stable. Furthermore it is easy to check that the Euler-Lagrange equations associated with the variational problem (4.25) coincide with equations (4.24) giving the neutrally stable states and, since the maximum of the functional $\mathcal{I}/\mathcal{D}$ is the reciprocal of the least positive eigenvalue of the characteristic value problem (4.24) with boundary conditions (4.21) or (4.22), the equality in (4.25) holds true. Therefore the modes of two-dimensional wave number a are linearly stable if and only if $R \le R_L(a)$. Next we introduce the so-called critical Rayleigh number

$$\mathscr{R}_c = \min_{a>0} R_L^2(a)^1, \tag{4.29}$$

and note that if $\mathscr{R} \le \mathscr{R}_c$ then all modes are stable, while if $\mathscr{R} > \mathscr{R}_c$ there exists at least one unstable mode. Thus the conduction solution m_0 is linearly stable if and only if $\mathscr{R} \le \mathscr{R}_c$.

[1] For any eigenfunction $(\bar{W},\bar{\theta})$ of the characteristic-value problem (4.24) with boundary

4.4 Nonlinear stability

Let now $\|\cdot\|$ denote the $L^2(\Omega)$ norm. In order to establish a nonlinear stability result we commence by multiplying $(4.10)_1$ by $\mathbf{u}$, $(4.10)_3$ by θ, and we then integrate over Ω to find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 = R\int_\Omega w\theta \mathrm{d}\Omega - 2\int_\Omega \mu(\bar{p}, \bar{T}+\theta)\mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega, \tag{4.30}$$

$$\frac{Pr}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|^2 = R\int_\Omega w\theta \mathrm{d}\Omega - \|\nabla\theta\|^2. \tag{4.31}$$

Hence by summing (4.30) and (4.31) we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 2R\int_\Omega w\theta \mathrm{d}\Omega - 2\int_\Omega \mu(\bar{p}, \bar{T}+\theta)\mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega - \|\nabla\theta\|^2, \tag{4.32}$$

where

$$E = \frac{1}{2}\|\mathbf{u}\|^2 + \frac{Pr}{2}\|\theta\|^2$$

is the sum of the kinetic and thermal energies associated with the perturbations.

We now state and prove a maximum principle (see Temam [89]) which will be very useful for our nonlinear stability analysis.

Lemma 4.1. *Let the disturbances* $\mathbf{u}$, P_1, θ *satisfy* (4.10) *with boundary conditions*

$$w = \theta = 0 \quad at\ z = 0, 1. \tag{4.33}$$

Then, if

$$|\theta(\mathbf{x}, 0)| \le \Theta_0 \quad a.e.\ \mathbf{x}\in\Omega \tag{4.34}$$

for constant $\Theta_0 \ge \dfrac{R}{Pr}$, *it follows that*

$$|\theta(\mathbf{x}, t)| \le \Theta_0 \quad a.e.\ \mathbf{x}\in\Omega, \quad a.e.\ t \ge 0.$$

Proof. We start by defining the truncation operators that associate with a function $\psi : \Omega \to \mathbb{R}$, the functions ψ_+ and ψ_-

$$\psi_+(\mathbf{x}) = \max\{\psi(\mathbf{x}), 0\}, \quad \psi_-(\mathbf{x}) = \max\{-\psi(\mathbf{x}), 0\}, \quad \mathbf{x}\in\Omega.$$

---

conditions (4.21) or (4.22) $\mathcal{I}(\bar{W},\bar{\theta})/\mathcal{D}(\bar{W},\bar{\theta})$ is a positive continuous function of the wave number a such that

$$\lim_{a\to 0^+}\frac{\mathcal{I}(\bar{W},\bar{\theta})}{\mathcal{D}(\bar{W},\bar{\theta})} = \lim_{a\to +\infty}\frac{\mathcal{I}(\bar{W},\bar{\theta})}{\mathcal{D}(\bar{W},\bar{\theta})} = 0,$$

then it admits maximum in $]0, +\infty[$. Consequently $R_L(a)$ is a positive continuous function such that $R_L(a) \to +\infty$ as $a \to 0^+$ and as $a \to +\infty$ and it admits minimum in $]0, +\infty[$.

Since $\mathbf{u}$, P_1, θ satisfy equations (4.10) with boundary conditions (4.33), the functions

$$T = T_2 + (T_1 - T_2)\left(1 - \frac{z}{d} + \frac{Pr}{R}\theta\right), \quad \mathbf{v} = \frac{\mu_0}{\rho_0 d}\mathbf{u},$$

$$P = \frac{\mu_0 P_1}{\alpha(T_1 - T_2)\rho_0 d^2} - \rho_0 g z\left(\frac{z}{2d} - 1\right) + P_0, \quad p = -\rho_0 g(z - d) + p_0$$

satisfy the boundary value problem (4.1)-(4.2).

We now prove that

$$T = \hat{T} + \tilde{T}$$

with

$$T_1^* = T_1 - (T_1 - T_2)\frac{Pr}{R}\Theta_0 \le \hat{T}(\mathbf{x}, t) \le T_2 + (T_1 - T_2)\frac{Pr}{R}\Theta_0 = T_2^*$$

almost everywhere in Ω for almost every $t \ge 0$ and $\tilde{T}(\cdot, t) \to 0$ in $\mathrm{L}^2(\Omega)$ as $t \to +\infty$.

Since $\theta \in \mathrm{H}^1(\Omega)$, it is clear that $(T - T_2^*)_+$ and $(T - T_1^*)_-$ are also in $\mathrm{H}^1(\Omega)$. Multiplying $(4.1)_4$ by $(T - T_2^*)_+$ and integrating over Ω we obtain, by taking into account the periodicity of the perturbations and the Poincaré inequality,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(T - T_2^*)_+\|^2 + \kappa\frac{\pi^2}{d^2}\|(T - T_2^*)_+\|^2 \le 0 \tag{4.35}$$

by which we deduce that $\|(T - T_2^*)_+(\cdot, t)\|$ decreases exponentially

$$\|(T - T_2^*)_+(\cdot, t)\| \le \|(T - T_2^*)_+(\cdot, 0)\| \exp\left(-\kappa\frac{\pi^2}{d^2}t\right).$$

Similarly we prove that

$$\|(T - T_1^*)_-(\cdot, t)\| \le \|(T - T_1^*)_-(\cdot, 0)\| \exp\left(-\kappa\frac{\pi^2}{d^2}t\right).$$

Thus, setting

$$\tilde{T} = (T - T_2^*)_+ - (T - T_1^*)_- \quad \text{and} \quad \hat{T} = T - \tilde{T},$$

we see that

$$T_1^* \le \hat{T}(\mathbf{x}, t) \le T_2^* \quad \text{a.e. } \mathbf{x} \in \Omega, \text{ a.e. } t \ge 0$$

and

$$\|\tilde{T}(\cdot, t)\| \le \{\|(T - T_1^*)_-\| + \|(T - T_2^*)_+\|\}_{t=0} \exp\left(-\kappa\frac{\pi^2}{d^2}t\right).$$

Then

$$\theta = \hat{\theta} + \tilde{\theta}$$

with

$$-\Theta_0 \le \hat{\theta} = \frac{R}{Pr(T_1 - T_2)}\left[\hat{T} - T_2 - (T_1 - T_2)\left(1 - \frac{z}{d}\right)\right] \le \Theta_0$$

almost everywhere in Ω for almost every $t \ge 0$, and

$$\tilde{\theta} = \frac{R}{Pr(T_1 - T_2)}\tilde{T}.$$

But

$$(\theta - \Theta_0)_+ = \tilde{\theta}_+ = \frac{R}{Pr(T_1 - T_2)}(T - T_2^*)_+$$

and

$$(\theta + \Theta_0)_- = \tilde{\theta}_- = \frac{R}{Pr(T_1 - T_2)}(T - T_1^*)_- \, .$$

Therefore

$$\begin{aligned}
\|(\theta - \Theta_0)_+(\cdot, t)\| &= \frac{R}{Pr(T_1 - T_2)}\|(T - T_2^*)_+(\cdot, t)\| \\
&\le \frac{R}{Pr(T_1 - T_2)}\|(T - T_2^*)_+(\cdot, 0)\| \exp\left(-\kappa\frac{\pi^2}{d^2}t\right) \\
&= \|(\theta - \Theta_0)_+(\cdot, 0)\| \exp\left(-\kappa\frac{\pi^2}{d^2}t\right)
\end{aligned}$$

and, similarly,

$$\|(\theta + \Theta_0)_-(\cdot, t)\| \le \|(\theta + \Theta_0)_-(\cdot, 0)\| \exp\left(-\kappa\frac{\pi^2}{d^2}t\right).$$

For (4.34) we observe that $\|(\theta - \Theta_0)_+(\cdot, t)\|$ and $\|(\theta + \Theta_0)_-(\cdot, t)\|$ are decreasing functions of time that vanish at $t = 0$ and, consequently, they vanish for all later time $t > 0$. Thus $\tilde{\theta} = 0$ and the proof is completed. □

As an immediate consequence of Lemma 4.1, if the initial disturbance to the temperature field $\bar{T}$ satisfies the inequality

$$|\theta_0(\mathbf{x})| \le \Theta_0 \quad \text{a.e. } \mathbf{x} \in \Omega \tag{4.36}$$

for $\Theta_0 \ge R/Pr$ such that, by the analyticity of μ, we can write

$$\mu(\bar{p}, \bar{T} + \theta_0) = \sum_{n=0}^{+\infty} \frac{1}{n!}\frac{\partial^n \mu}{\partial T^n}(\bar{p}, \bar{T})\theta_0^n \quad \text{a.e. } \mathbf{x} \in \Omega,$$

then

$$|\theta(\mathbf{x},t)| \le \Theta_0 \quad \text{a.e. } \mathbf{x}\in\Omega, \quad \text{a.e. } t \ge 0 \tag{4.37}$$

and further

$$\mu(\bar{p},\bar{T}+\theta) = \sum_{n=0}^{+\infty} \frac{1}{n!}\frac{\partial^n \mu}{\partial T^n}(\bar{p},\bar{T})\theta^n \quad \text{a.e. } \mathbf{x}\in\Omega, \quad \text{a.e. } t \ge 0. \tag{4.38}$$

Therefore for initial thermal disturbances satisfying (4.36), by (4.37) and (4.38), we have

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= -\left(1 - R\frac{\mathscr{I}}{\mathscr{D}}\right)\mathscr{D} - 2\int_\Omega \sum_{n=1}^{+\infty} \frac{1}{n!}\frac{\partial^n \mu}{\partial T^n}(\bar{p},\bar{T})\theta^n \mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega \\ &\le -\left[1 - \frac{R}{R_E(a)}\right]\mathscr{D} + 2\int_\Omega \sum_{n=1}^{+\infty} \frac{1}{n!}\left|\frac{\partial^n \mu}{\partial T^n}(z)\right|\Theta_0^n \mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega \\ &\le -\left[1 - \frac{R}{R_E(a)}\right]\mathscr{D} + 2M\|\mathbf{d}\|^2,\end{aligned} \tag{4.39}$$

where

$$\begin{aligned}\mathscr{I} &= 2\int_\Omega w\theta \mathrm{d}\Omega, \\ \mathscr{D} &= 2\int_\Omega \mu(z)\mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega + \|\nabla\theta\|^2, \\ \frac{1}{R_E(a)} &= \max_{\mathscr{W}} \frac{\mathscr{I}}{\mathscr{D}},\end{aligned} \tag{4.40}$$

$\mathscr{W}$ being the set of the kinematically admissible fields:

$$\begin{aligned}\mathscr{W} = \big\{(\mathbf{u},\theta) \in \left(\mathrm{H}^1(\Omega)\right)^4 : \mathbf{u},\ \theta \text{ periodic in } x \text{ and } y \text{ of periods} \\ 2\pi/a_x,\ 2\pi/a_y, \operatorname{div}\mathbf{u} = 0,\ \ \mathbf{u} = \mathbf{0} \text{ and } \theta = 0 \text{ at } z = 0,1\big\}\end{aligned}$$

for rigid boundary conditions,

$$\begin{aligned}\mathscr{W} = \big\{(\mathbf{u},\theta) \in \left(\mathrm{H}^1(\Omega)\right)^4 : \mathbf{u},\ \theta \text{ periodic in } x \text{ and } y \text{ of periods } 2\pi/a_x,\ 2\pi/a_y, \\ \mathbf{u} \text{ satisfies (4.11)},\ \operatorname{div}\mathbf{u} = 0,\ \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = \theta = 0 \text{ at } z = 0,1\big\}\end{aligned}$$

for stress-free boundary conditions, and

$$M = \max_{z\in[0,1]} \sum_{n=1}^{+\infty} \frac{1}{n!}\left|\frac{\partial^n \mu}{\partial T^n}(z)\right|\Theta_0^n.$$

By following Rionero [73] we prove the existence of the maximum of the functional $\mathscr{I}/\mathscr{D}$ in $\mathscr{W}$.

Theorem 4.2. *The functional $\mathscr{I}/\mathscr{D}$ admits maximum in $\mathscr{W}$.*

Proof. Taking into account the periodicity and the boundary conditions, by Poincaré and Wirtinger inequalities we have

$$\begin{aligned}\mathscr{D}(\mathbf{u},\theta) &= 2\int_\Omega \mu(z)\mathbf{d}\cdot\mathbf{d}\mathrm{d}\Omega + \|\nabla\theta\|^2 \geq \mu_{\min}\|\nabla\mathbf{u}\|^2 + \|\nabla\theta\|^2 \\ &\geq \mu_{\min}\pi_0^2\|\mathbf{u}\|^2 + \pi^2\|\theta\|^2 \quad \forall(\mathbf{u},\theta)\in\mathscr{W},\end{aligned}$$

where

$$\mu_{\min} = \min_{z\in[0,1]}\mu(z), \quad \mu_{\max} = \max_{z\in[0,1]}\mu(z) \quad \text{and} \quad \pi_0^2 = \min\{a_x^2, a_y^2, \pi^2\}.$$

Then, by Cauchy inequality, the functional $\mathscr{I}/\mathscr{D}$ is bounded from above

$$\begin{aligned}\frac{\mathscr{I}(\mathbf{u},\theta)}{\mathscr{D}(\mathbf{u},\theta)} &\leq \frac{\|\mathbf{u}\|^2+\|\theta\|^2}{\mu_{\min}\pi_0^2\|\mathbf{u}\|^2+\pi^2\|\theta\|^2} \\ &\leq \max\left\{\frac{1}{\mu_{\min}\pi_0^2}, \frac{1}{\pi^2}\right\} \quad \forall(\mathbf{u},\theta)\in\mathscr{W}.\end{aligned}$$

Let now

$$\frac{1}{R_E(a)} = \sup_{(\mathbf{u},\theta)\in\mathscr{W}} \frac{\mathscr{I}(\mathbf{u},\theta)}{\mathscr{D}(\mathbf{u},\theta)} \tag{4.41}$$

and

$$\{\mathbf{u}_n,\theta_n\}_{n\in\mathbb{N}} \subset \mathscr{W}, \quad \mathscr{D}(\mathbf{u}_n,\theta_n) = 1 \ \forall n\in\mathbb{N}$$

be a maximizing sequence, viz

$$\lim_{n\to+\infty}\mathscr{I}(\mathbf{u}_n,\theta_n) = \frac{1}{R_E(a)}. \tag{4.42}$$

We now observe that since $\mathscr{W}$ is a closed linear subspace of $(\mathrm{H}^1(\Omega))^4$ it is also weakly closed (see [50] page 134). Furthermore

$$\begin{aligned}\frac{1}{2}\min\left\{\mu_{\min}\pi_0^2, \mu_{\min}, 1\right\}\|(\mathbf{u},\theta)\|_{\mathscr{W}}^2 &\leq \mathscr{D}(\mathbf{u},\theta) \\ &\leq \max\left\{\mu_{\max}, 1\right\}\|(\mathbf{u},\theta)\|_{\mathscr{W}}^2 \quad \forall(\mathbf{u},\theta)\in\mathscr{W},\end{aligned} \tag{4.43}$$

that is in $\mathscr{W}$ the positive definite functional $\mathscr{D}$ defines a norm which is equivalent to that induced by the standard $(\mathrm{H}^1(\Omega))^4$-norm

$$\|(\mathbf{u},\theta)\|_{\mathscr{W}} = \left(\|\mathbf{u}\|^2+\|\nabla\mathbf{u}\|^2+\|\theta\|^2+\|\nabla\theta\|^2\right)^{1/2}.$$

Therefore by the previous observations and by Rellich-Kondrachov compact embedding Theorem there exists $(\mathbf{u}^*,\theta^*)\in\mathscr{W}$ such that, except for subsequences,

$$(\mathbf{u}_n,\theta_n) \rightharpoonup (\mathbf{u}^*,\theta^*) \quad \text{weakly in } (\mathrm{H}^1(\Omega))^4$$

and

$$(\mathbf{u}_n, \theta_n) \to (\mathbf{u}^*, \theta^*) \quad \text{strongly in } (\mathrm{L}^2(\Omega))^4. \tag{4.44}$$

We now prove that $\{(\mathbf{u}_n, \theta_n)\}_n$ is a Cauchy sequence with respect to the norm defined by $\mathscr{D}$ in $\mathscr{W}$. Obviously

$$\begin{cases} \mathscr{I}\left(\dfrac{\mathbf{u}_n + \mathbf{u}_m}{2}, \dfrac{\theta_n + \theta_m}{2}\right) = \frac{1}{2}\mathscr{I}(\mathbf{u}_n, \theta_n) + \frac{1}{2}\mathscr{I}(\mathbf{u}_m, \theta_m) \\ \qquad\qquad -\mathscr{I}\left(\dfrac{\mathbf{u}_n - \mathbf{u}_m}{2}, \dfrac{\theta_n - \theta_m}{2}\right) \\ \mathscr{D}\left(\dfrac{\mathbf{u}_n - \mathbf{u}_m}{2}, \dfrac{\theta_n - \theta_m}{2}\right) = \frac{1}{2}\mathscr{D}(\mathbf{u}_n, \theta_n) + \frac{1}{2}\mathscr{D}(\mathbf{u}_m, \theta_m) \\ \qquad\qquad -\mathscr{D}\left(\dfrac{\mathbf{u}_n + \mathbf{u}_m}{2}, \dfrac{\theta_n + \theta_m}{2}\right). \end{cases} \tag{4.45}$$

Let $\epsilon > 0$. By (4.41), (4.42) and (4.45) there exists $\nu_\epsilon \in \mathbb{N}$ such that

$$\frac{1}{R_E(a)}\left(1 - \frac{\epsilon}{8}\right) < \mathscr{I}(\mathbf{u}_n, \theta_n) < \frac{1}{R_E(a)}\left(1 + \frac{\epsilon}{8}\right) \quad \forall n \geq \nu_\epsilon,$$

$$\begin{aligned} \mathscr{D}\left(\frac{\mathbf{u}_n + \mathbf{u}_m}{2}, \frac{\theta_n + \theta_m}{2}\right) &\geq R_E(a)\mathscr{I}\left(\frac{\mathbf{u}_n + \mathbf{u}_m}{2}, \frac{\theta_n + \theta_m}{2}\right) \\ &> 1 - \frac{\epsilon}{8} - R_E(a)\mathscr{I}\left(\frac{\mathbf{u}_n - \mathbf{u}_m}{2}, \frac{\theta_n - \theta_m}{2}\right) \quad \forall n, m \geq \nu_\epsilon \end{aligned}$$

and

$$\mathscr{D}\left(\frac{\mathbf{u}_n - \mathbf{u}_m}{2}, \frac{\theta_n - \theta_m}{2}\right) < \frac{\epsilon}{8} + R_E(a)\mathscr{I}\left(\frac{\mathbf{u}_n - \mathbf{u}_m}{2}, \frac{\theta_n - \theta_m}{2}\right) \quad \forall n, m \geq \nu_\epsilon.$$

On the other hand by Hölder inequality and (4.44) there exists $\nu'_\epsilon \in \mathbb{N}$ such that

$$\mathscr{I}\left(\frac{\mathbf{u}_n - \mathbf{u}_m}{2}, \frac{\theta_n - \theta_m}{2}\right) < \frac{\epsilon}{8R_E(a)} \quad \forall n, m \geq \nu'_\epsilon,$$

and hence

$$\mathscr{D}(\mathbf{u}_n - \mathbf{u}_m, \theta_n - \theta_m) < \epsilon \quad \forall n, m \geq \max\{\nu_\epsilon, \nu'_\epsilon\}.$$

Since the norm defined by $\mathscr{D}$ in $\mathscr{W}$ is equivalent to $\|(\cdot,\cdot)\|_{\mathscr{W}}$ and since $(\mathscr{W}, \|(\cdot,\cdot)\|_{\mathscr{W}})$ is a Banach space, $(\mathbf{u}_n, \theta_n)$ converges strongly to $(\mathbf{u}_*, \theta_*)$ in $(\mathrm{H}^1(\Omega))^4$ and $\mathscr{D}(\mathbf{u}_*, \theta_*) = 1$.

Finally (4.44) and the continuity of the functional $\mathscr{I}$ in $(\mathrm{L}^2(\Omega))^4$ yield

$$\frac{1}{R_E(a)} = \lim_{n \to +\infty} \mathscr{I}(\mathbf{u}_n, \theta_n) = \mathscr{I}(\mathbf{u}^*, \theta^*).$$

The proof is thus completed. □

By assuming
$$R < R_E(a)$$
and by choosing Θ_0 such that
$$M < \left[1 - \frac{R}{R_E(a)}\right] \mu_{\min},$$
from (4.39), by Poincaré and Wirtinger inequalities, we deduce the following energy inequality
$$\frac{\mathrm{d}E}{\mathrm{d}t} \le -\left[1 - \frac{R}{R_E(a)}\right] \nu_a E(t) \tag{4.46}$$
where
$$\nu_a = 2\min\left\{\pi_0^2\left[\mu_{\min} - \frac{MR_E(a)}{R_E(a) - R}\right], \frac{\pi^2}{Pr}\right\}.$$
Integrating (4.46) we have
$$E(t) \le E(0)\exp\left\{-\left[1 - \frac{R}{R_E(a)}\right]\nu_a t\right\}. \tag{4.47}$$

The number $R_E(a)$ is found from the variational problem (4.40) and the Euler-Lagrange equations corresponding to this are
$$\begin{cases} -\nabla\chi = \mu(z)\Delta\mathbf{u} + \mu'(z)\Big[\Big(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\Big)\mathbf{i} \\ \qquad + \Big(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\Big)\mathbf{j} + 2\dfrac{\partial w}{\partial z}\mathbf{k}\Big] + R\theta\mathbf{k} \\ \operatorname{div}\mathbf{u} = 0 \\ \Delta\theta + Rw = 0, \end{cases} \tag{4.48}$$
where χ is a Lagrange multiplier associated with the divergence constraint. This eigenvalue problem is exactly the same as the one of linear stability theory and hence the critical Rayleigh numbers for the linear and nonlinear stability problems coincide. Finally, by Lemma 4.1 and by (4.47) we may state the following

Theorem 4.3. *Assume that*
$$\mathscr{R} < \mathscr{R}_c$$
with $\mathscr{R}_c$ *given by* (4.29)*, and*
$$|\theta_0(\mathbf{x})| \le \Theta_0 \quad a.e.\ \mathbf{x} \in \mathbb{R}^2 \times [0,1]$$
for constant $\Theta_0 \ge R/Pr$ *such that*
$$M = \max_{z\in[0,1]} \sum_{n=1}^{+\infty} \frac{1}{n!}\left|\frac{\partial^n \mu}{\partial T^n}(z)\right| \Theta_0^n < \left(1 - \sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}}\right)\mu_{\min}.$$

Then the conduction solution m_0 is nonlinearly stable with respect to the energy of the perturbations E(t), and

$$E(t) \leq E(0) \exp \left[- \left(1 - \sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}} \right) \nu t \right],$$

where

$$\nu = 2 \min \left\{ \pi_0^2 \left[\mu_{\min} - M \left(1 - \sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}} \right)^{-1} \right], \frac{\pi^2}{Pr} \right\}.$$

Remark 4.1. For temperature-dependent viscous fluids studied by Capone and Gentile in [8] and by Richardson and Straughan in [72] it is was found that the critical Rayleigh number depends on the choice of the reference temperature. Here we observe that the critical Rayleigh number will obviously depend on the choice of the reference pressure as well as of the reference temperature since the function $\mu(z)$ defined in (4.12) varies according to the choice of the reference state. We have chosen the values of pressure and temperature at the top of the fluid layer as reference state because we think this choice could be more convenient in the practical applications.

Remark 4.2. The results of this chapter may be condensed in the sentence: by using the generalization of the Oberbeck-Boussinesq equations we have derived in section 2.4 which is valid at small values of the dimensionless quantity $\alpha(T_1 - T_2)$, the nonlinear energy stability result agrees with the linear one even when viscosity is an analytic function of both temperature and pressure.

4.5 Numerical results

We now consider the pressure-temperature-viscosity relationship (3.12), nondimensionalize it as indicated in (4.9) and obtain the dimensionless viscosity

$$\mu(z) = \exp[\Gamma(z-1)]$$

with $\Gamma = \gamma(T_1 - T_2) - \beta\rho_0 g d$. The equations giving the marginal stable disturbances are then

$$\begin{cases} 2\Gamma \mathrm{D}(\mathrm{D}^2 - a^2)W + (\mathrm{D}^2 - a^2)^2 W \\ \qquad\qquad + \Gamma^2(\mathrm{D}^2 + a^2)W = \exp[-\Gamma(z-1)] R a^2 \Theta \\ (\mathrm{D}^2 - a^2)\Theta + RW = 0 \end{cases}$$

with boundary conditions (4.21) or (4.22). By employing the Galerkin-type method developed by Chandrasekhar [11] we find approximations to the critical Rayleigh number for different values of the dimensionless parameter Γ both for rigid (Table 4.1) and free (Table 4.2) bounding surfaces. For rigid boundaries we used "beam functions" (see [26]) and sines for free surfaces. We observe that for $\Gamma = 0$, in particular for constant viscosity ($\beta = 0$ and $\gamma = 0$), we obtain the classical results (see for instance [17]).

Table 4.1: Approximations to the critical Rayleigh and wave numbers against Γ in the rigid case.

Γ	$\mathscr{R}_c$	a_c
-2	5026.42	3.072
-1.5	3790.86	3.084
-1	2885.93	3.093
-0.5	2217.33	3.100
0	1707.76	3.117
0.5	1344.88	3.100
1	1061.67	3.093
1.5	845.855	3.084
2	680.252	3.072

Table 4.2: Approximations to the critical Rayleigh and wave numbers against Γ in the stress-free case.

Γ	$\mathscr{R}_c$	a_c
-2	1991.74	2.134
-1.5	1480.22	2.171
-1	1114.29	2.198
-0.5	850.114	2.216
0	657.51	2.221
0.5	515.62	2.216
1	409.926	2.198
1.5	330.281	2.171
2	269.552	2.134

As concerns the nonlinear energy stability analysis, Theorem 4.3 may be re-stated as follows

Theorem 4.4. *Assume that*

$$\mathscr{R} < \mathscr{R}_c$$

with $\mathscr{R}_c$ given by (4.29),

$$\gamma(T_1 - T_2) < \ln\left\{1 + \left(1 - \sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}}\right)\exp(-|\Gamma|)\right\}$$

and

$$|\theta_0(\mathbf{x})| \leq \Theta_0 \in \left[\frac{\sqrt{\mathscr{R}}}{Pr}, \frac{\sqrt{\mathscr{R}}}{\gamma(T_1 - T_2)Pr}\ln\left\{1 + \left(1 - \sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}}\right)\exp(-|\Gamma|)\right\}\right[$$

almost everywhere in $\mathbb{R}^2 \times [0,1]$. *Then the conduction solution m_0 is nonlinearly stable with respect to the energy of the perturbations E(t), and*

$$E(t) \leq E(0)\exp\left[-\left(1 - \sqrt{\frac{\mathscr{R}}{\mathscr{R}_c}}\right)\nu t\right],$$

where

$$\nu = 2\min\left\{\pi_0^2 A, \frac{\pi^2}{Pr}\right\},$$

$$A = \begin{cases} \exp(-\Gamma) - \dfrac{\exp\left[\dfrac{\gamma(T_1 - T_2)Pr}{\sqrt{\mathscr{R}}}\Theta_0\right] - 1}{1 - \sqrt{\dfrac{\mathscr{R}}{\mathscr{R}_c}}} & \text{if } \Gamma \geq 0 \\[2em] 1 - \exp(-\Gamma)\dfrac{\exp\left[\dfrac{\gamma(T_1 - T_2)Pr}{\sqrt{\mathscr{R}}}\Theta_0\right] - 1}{1 - \sqrt{\dfrac{\mathscr{R}}{\mathscr{R}_c}}} & \text{if } \Gamma < 0. \end{cases}$$

Chapter 5

Stability of MHD laminar flows in a porous medium with Brinkman law

5.1 Mathematical formulation of the problem

This chapter is devoted to the study of the stability of the laminar flows in a homogeneous, incompressible, electrically conducting fluid saturating an infinite horizontal porous layer embedded in a constant magnetic field. This problem has been studied by Rudraiah and Mariyappa in [78] [1] in order to investigate the effect of the geomagnetic field on non-convective flows in the geothermal region. It is known that in the geothermal region the sub-surface ground water posseses a general upward convective drift due to buoyancy induced by the high underground temperature. Since the rising ground water is cooled as it approaches the surface, where heat is removed by evaporation, radiation and movement in the surface streams, an unstable state may be induced and complicated convective motions appear in the layers near the surface. In those circumstances it is of practical interest to consider the effect of the geomagnetic field on such flows and see whether the magnetic field inhibits this instability. In particular in [78] Rudraiah and Mariyappa studied the stability of steady hydromagnetic flows in a porous medium by assuming the fluid with a finite electrical conductivity, valid the Oberbeck-Boussinesq approximation and neglecting the effects of its viscosity with respect to the friction that manifests itself at the pores. Here, instead, we include the frictional forces in the fluid by considering the unsteady Brinkman model for flows of a viscous fluid in a porous medium.

[1] The stability of MHD laminar flows, in different situations, is also studied in [42, 43, 74].

Let $Oxyz$ be a cartesian frame of reference with unit vector fields $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, respectively, $\mathbf{k}$ pointed vertically upward, and let $\Omega_d = \mathbb{R}^2 \times (-d, d)$ ($d = \text{const} > 0$) be a horizontal porous layer bounded by the planes $z = \pm d$, assumed both rigid, electrically non conducting and at rest or in motion with velocity parallel to the plane $z = 0$. As the fluid filling Ω_d is concerned we assume that it is homogeneous, electrically conducting, embedded in a constant magnetic field $\mathbf{H}_0 = H_0\mathbf{k}$ and submitted to a conservative force $\mathbf{F}$ with potential $\mathcal{U}$. By following the same arguments in sections 2.5 and 2.8, the equations of non-relativistic magnetohydrodynamics in a porous medium in the isothermal case are the usual equations governing the fluid flow in a porous matrix suitably modified to take into account of the Lorentz force, to which equations $(2.63)_4$ and $(2.63)_5$ are added:

$$\begin{cases} \frac{\rho_0}{\varphi}\mathbf{v}_t = -\nabla P - \frac{\mu_1}{\varphi K}\mathbf{v} + \frac{\mu_2}{\varphi}\Delta\mathbf{v} + \mu_m(H_0\mathbf{k} + \mathbf{H}) \cdot \nabla\mathbf{H} \\ \operatorname{div}\mathbf{v} = \operatorname{div}\mathbf{H} = 0 \\ \mathbf{H}_t + \nabla \times [(H_0\mathbf{k} + \mathbf{H}) \times \mathbf{v}] = \eta\Delta\mathbf{H} \end{cases} \tag{5.1}$$

where

$\mathbf{v} = (U, V, W)$ the seepage velocity,	ρ_0 the density of the fluid,
$\mathbf{H}$ the induced magnetic field,	μ_m the magnetic permeability,
μ_i $(i = 1, 2)$ viscosity coefficients,	η the magnetic viscosity of the fluid,
φ the porosity of the medium,	K the permeability of the medium

and

$$P = p + \frac{\mu_m}{2}|H_0\mathbf{k} + \mathbf{H}|^2 - \rho_0\mathcal{U}$$

is the generalized pressure.

To equations (5.1) we append the boundary conditions

$$\begin{cases} U(x, y, -d, t) = U_1(x, y, t), \quad U(x, y, d, t) = U_2(x, y, t) \\ V(x, y, -d, t) = V_1(x, y, t), \quad V(x, y, d, t) = V_2(x, y, t) \\ W(x, y, -d, t) = W(x, y, d, t) = 0 \\ \mathbf{H}(x, y, -d, t) = \mathbf{H}(x, y, d, t) = \mathbf{0} \end{cases} \tag{5.2}$$

with U_i, V_i $(i = 1, 2)$ assigned regular fields on $\mathbb{R}^2 \times [0. + \infty)$. Boundary conditions $(5.2)_1$-$(5.2)_3$ tell us that the fluid adheres to the impermeable plates $z = \pm d$ whereas $(5.2)_4$ yields that the magnetic field is continuous at the boundaries as the bounding surfaces are electrically non-conducting [1].

In order to non-dimensionalize equations (5.1) we introduce the following non-dimensional quantities

$$\mathbf{x}^* = \frac{\mathbf{x}}{d}, \ t^* = \frac{\mu_2}{\rho_0 d^2}t, \ \mathbf{v}^* = \frac{\rho_0 d}{\mu_2}\mathbf{v}, \ \mathbf{H}^* = \frac{\mathbf{H}}{H_0}, \ P^* = \frac{\varphi\rho_0 K}{\mu_1\mu_2}P,$$

substitute them in (5.1) and get the dimensionless equations governing the motion (omitting the asterisks)

$$\begin{cases} \tilde{D}a\mathbf{v}_t = -\nabla P - \mathbf{v} + \tilde{D}a\Delta\mathbf{v} + \tilde{D}aP_m(\mathbf{H}\cdot\nabla\mathbf{H} + \mathbf{H}_z) \\ \operatorname{div}\mathbf{v} = \operatorname{div}\mathbf{H} = 0 \\ \mathbf{H}_t + \nabla\times[(\mathbf{k}+\mathbf{H})\times\mathbf{v}] = \dfrac{1}{R_m}\Delta\mathbf{H} \end{cases} \tag{5.3}$$

where

$$\tilde{D}a = \frac{K\mu_2}{d^2\mu_1},\ \ P_m = \frac{\varphi\mu_m H_0^2 d^2\rho_0}{\mu_2^2},\ \ R_m = \frac{\mu_2}{\varphi\eta\rho_0},\ \ M = \sqrt{P_m R_m}$$

are, respectively, the Darcy, the magnetic pressure, the magnetic Reynolds and the Hartmann numbers. Of course, the initial and the boundary conditions must be modified according to the chosen scalings.

5.2 Laminar MHD flows

Looking for the one-dimensional laminar flows of the type

$$\begin{cases} \mathbf{v} = U(z)\mathbf{i} \\ \mathbf{H} = H(z)\mathbf{i}, \end{cases}$$

from (5.1), it turns out that (U, H) have to fulfil the following system:

$$\begin{cases} \nabla P = \left(-U + \tilde{D}a\dfrac{\mathrm{d}^2U}{\mathrm{d}z^2} + \tilde{D}aP_m\dfrac{\mathrm{d}H}{\mathrm{d}z}\right)\mathbf{i} \\ \dfrac{\mathrm{d}^2H}{\mathrm{d}z^2} + R_m\dfrac{\mathrm{d}U}{\mathrm{d}z} = 0 \end{cases} \tag{5.4}$$

with boundary conditions

$$\begin{cases} U(-1) = U_1, \quad U(1) = U_2 \\ H(\pm 1) = 0 \end{cases} \tag{5.5}$$

where U_1 and U_2 are assigned constants. It is easy to check that the solutions of the boundary value problem (5.4)-(5.5) are given by

$$\begin{cases} U(z) = A_1\mathrm{e}^{\tau z} + A_2\mathrm{e}^{-\tau z} - \dfrac{A_0 + \tilde{D}aP_mB_1}{\tau^2\tilde{D}a} \\ H(z) = -R_m\left[\dfrac{A_1\mathrm{e}^{\tau z} - A_2\mathrm{e}^{-\tau z}}{\tau} - \dfrac{A_0 z}{\tau^2\tilde{D}a}\right] - \dfrac{B_1 z}{\tau^2\tilde{D}a} + B_2 \\ P = A_0 x + p_0 \end{cases} \tag{5.6}$$

where

$$\begin{cases} \tau = \sqrt{\dfrac{1+\tilde{D}aM^2}{\tilde{D}a}} \\ A_1 = \dfrac{U_2-U_1}{4\sinh\tau} + \dfrac{(U_1+U_2+2A_0)\tau}{4(\tau\cosh\tau + \tilde{D}aM^2\sinh\tau)} \\ A_2 = -\dfrac{U_2-U_1}{4\sinh\tau} + \dfrac{(U_1+U_2+2A_0)\tau}{4(\tau\cosh\tau + \tilde{D}aM^2\sinh\tau)} \\ B_1 = R_m\dfrac{2A_0\tau\cosh\tau - [\tau^2\tilde{D}a(U_1+U_2)+2A_0]\sinh\tau}{2(\tau\cosh\tau + \tilde{D}aM^2\sinh\tau)} \\ B_2 = \dfrac{R_m(U_2-U_1)}{2\tau}\coth\tau, \end{cases}$$

A_0 and p_0 are real constants.

5.2.1 Hartmann flow

As special case of (5.6), for $U(\pm 1) = 0$ and $A_0 \neq 0$, one obtains the Hartmann flow

$$\begin{cases} U(z) = \dfrac{A_0\tau\cosh\tau}{\tau\cosh\tau + \tilde{D}aM^2\sinh\tau}\left[\dfrac{\cosh(\tau z)}{\cosh\tau} - 1\right] \\ H(z) = \dfrac{R_m A_0\sinh\tau}{\tau\cosh\tau + \tilde{D}aM^2\sinh\tau}\left[z - \dfrac{\sinh(\tau z)}{\sinh\tau}\right] \\ P = A_0 x + p_0. \end{cases} \tag{5.7}$$

In Figure 5.1 we have plotted the normalized velocity profiles in Hartmann flow for different values of the parameter τ. The velocity U has been normalized by dividing $(5.7)_1$ by the velocity at the centre of the channel

$$V = \frac{A_0\tau(1-\cosh\tau)}{\tau\cosh\tau + \tilde{D}aM^2\sinh\tau}.$$

If τ is small, viscosity dominates the induction drag and the velocity profile is nearly parabolic. If τ is large, on the other hand, viscosity is unimportant save in thin boundary layers (thickness $\sim 1/\tau$) near the walls; away from the walls U is nearly constant. Concerning the induced magnetic field, from $(5.7)_2$ we deduce that it tends to zero as $\tau \to +\infty$ and then, if the magnetic Reynolds number is small, the embedding magnetic field lines are not greatly destorted by the flow.

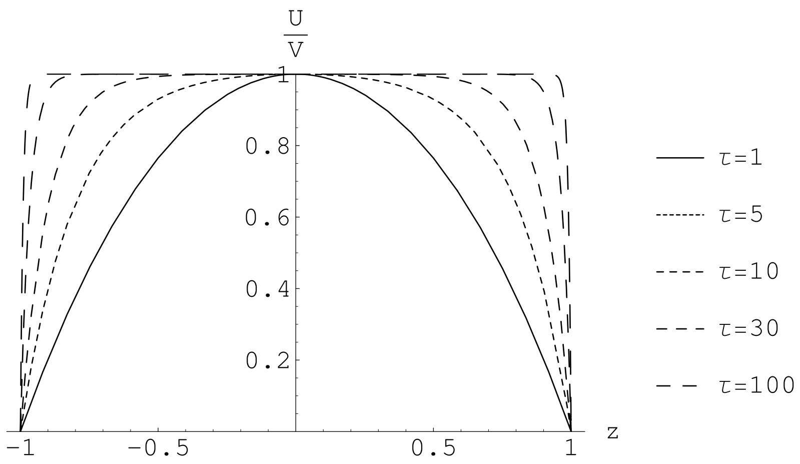

Figure 5.1: Normalized velocity profiles in Hartmann flow for different values of τ.

5.2.2 Couette flow

For $A_0 = 0$, $U(-1) = -V$ and $U(1) = V$, with $V = const$, the magnetic Couette flow is obtained

$$\begin{cases} U(z) = \dfrac{V \sinh(\tau z)}{\sinh \tau} \\ H(z) = \dfrac{R_m V \coth \tau}{\tau} \left[1 - \dfrac{\cosh(\tau z)}{\cosh \tau} \right] \\ P = p_0. \end{cases} \tag{5.8}$$

Normalized velocity profiles in Couette flow for different values of the parameter τ are shown in Figure 5.2. Velocity has been normalized by dividing $(5.8)_1$ by the velocity V of the upper plate.

If τ is small the velocity profile is nearly linear, while, if it is large, the seepage velocity is constant except for thin boundary layers (thickness $\sim 1/\tau$) near the walls where viscosity dominates the induction drag. From $(5.8)_2$, in the limit as $\tau \to +\infty$, the induced magnetic field tends to zero as in Hartmann flow.

Finally it is interesting to observe that for $M \to 0$, $(5.7)_1$ and $(5.8)_1$ give the Poiseuille and Couette flows found by Kaviany in [32].

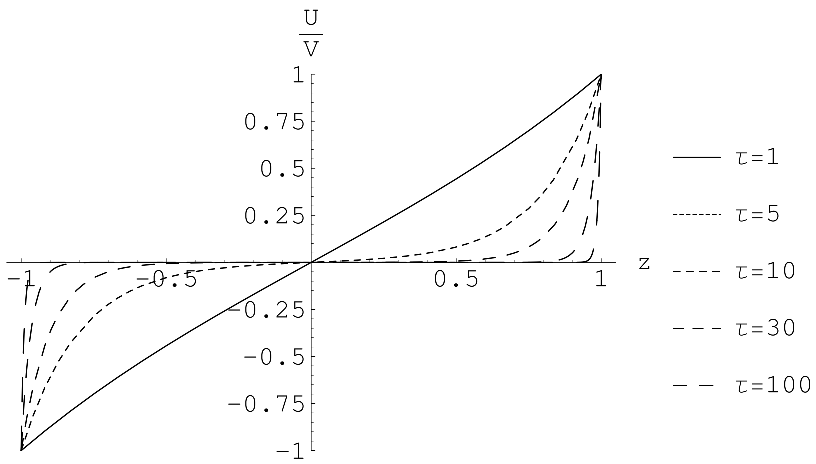

Figure 5.2: Normalized velocity profiles in Couette flow for different values of τ.

5.3 Sufficient condition for linear stability

The evolution equations of a perturbation $(\mathbf{u}, \mathbf{h}, p_1)$ to the basic laminar flow $m_0 = (\mathbf{v}, \mathbf{H}, P)$ given by (5.6) are

$$\begin{cases} \tilde{D}a\mathbf{u}_t = -\nabla p_1 - \mathbf{u} + \tilde{D}a\Delta\mathbf{u} \\ \qquad +\tilde{D}aP_m\left[\mathbf{h}\cdot\nabla\mathbf{h} + H'(z)h_3\mathbf{i} + H(z)\mathbf{h}_x + \mathbf{h}_z\right] \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{h} = 0 \\ \mathbf{h}_t = -\mathbf{u}\cdot\nabla\mathbf{h} - H'(z)u_3\mathbf{i} + \mathbf{h}\cdot\nabla\mathbf{u} + H(z)\mathbf{u}_x \\ \qquad +\mathbf{u}_z - U(z)\mathbf{h}_x + U'(z)h_3\mathbf{i} + \dfrac{1}{R_m}\Delta\mathbf{h} \end{cases} \tag{5.9}$$

in $\mathbb{R}^2 \times (-1, 1) \times (0, +\infty)$, under the initial and boundary conditions:

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{h}(\mathbf{x}, 0) = \mathbf{h}_0(\mathbf{x})$$

$$\mathbf{u}(x, y, \pm 1, t) = \mathbf{h}(x, y, \pm 1, t) = \mathbf{0}, \tag{5.10}$$

with $\mathbf{u}_0$, $\mathbf{h}_0$ assigned divergence-free regular fields.

Linearizing (5.9) we obtain

$$\begin{cases} \tilde{D}a\mathbf{u}_t = -\nabla p_1 - \mathbf{u} + \tilde{D}a\Delta\mathbf{u} + \tilde{D}aP_m\left[H'(z)h_3\mathbf{i} + H(z)\mathbf{h}_x + \mathbf{h}_z\right] \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{h} = 0 \\ \mathbf{h}_t = -H'(z)u_3\mathbf{i} + H(z)\mathbf{u}_x + \mathbf{u}_z - U(z)\mathbf{h}_x + U'(z)h_3\mathbf{i} + \dfrac{1}{R_m}\Delta\mathbf{h} \end{cases} \tag{5.11}$$

and, as in section 4.3, look for periodic solutions in the x, y directions of periods $2\pi/a_x$ and $2\pi/a_y$ ($a_x > 0$, $a_y > 0$), respectively,

$$\begin{cases} \mathbf{u}'(x,y,z,t) = \hat{\mathbf{u}}(z)\exp[\mathrm{i}(a_x x + a_y y) + ct] \\ \mathbf{h}'(x,y,z,t) = \hat{\mathbf{h}}(z)\exp[\mathrm{i}(a_x x + a_y y) + ct] \\ p_1'(x,y,z,t) = \hat{p}_1(z)\exp[\mathrm{i}(a_x x + a_y y) + ct] \end{cases} \tag{5.12}$$

with complex wave speed $c = c_r + \mathrm{i}c_i$ and two-dimensional wave number $a = (a_x^2 + a_y^2)^{1/2}$.
Denoting by D the differential operator $\mathrm{d}/\mathrm{d}z$, we substitute the expressions (5.12) into (5.11) and obtain the following system of ordinary differential equations:

$$\begin{cases} \tilde{D}ac\hat{u}_1 + \hat{u}_1 - \tilde{D}a(\mathrm{D}^2 - a^2)\hat{u}_1 - \tilde{D}aP_m\big[\mathrm{D}\hat{h}_1 \\ \qquad +\mathrm{i}a_x H(z)\hat{h}_1 + H'(z)\hat{h}_3\big] = -\mathrm{i}a_x\hat{p}_1 \\ \tilde{D}ac\hat{u}_2 + \hat{u}_2 - \tilde{D}a(\mathrm{D}^2 - a^2)\hat{u}_2 - \tilde{D}aP_m(\mathrm{D}\hat{h}_2 \\ \qquad +\mathrm{i}a_x H(z)\hat{h}_2) = -\mathrm{i}a_y\hat{p}_1 \\ \tilde{D}ac\hat{u}_3 + \hat{u}_3 - \tilde{D}a(\mathrm{D}^2 - a^2)\hat{u}_3 - \tilde{D}aP_m\big[\mathrm{D}\hat{h}_3 \\ \qquad +\mathrm{i}a_x H(z)\hat{h}_3\big] = \mathrm{D}\hat{p}_1 \\ \mathrm{D}\hat{u}_3 = -\mathrm{i}a_x\hat{u}_1 - \mathrm{i}a_y\hat{u}_2 \\ \mathrm{D}\hat{h}_3 = -\mathrm{i}a_x\hat{h}_1 - \mathrm{i}a_y\hat{h}_2 \\ c\hat{h}_1 + \mathrm{i}a_x U(z)\hat{h}_1 + H'(z)\hat{u}_3 - U'(z)\hat{h}_3 - \mathrm{D}\hat{u}_1 \\ \qquad -\mathrm{i}a_x H(z)\hat{u}_1 - R_m^{-1}(\mathrm{D}^2 - a^2)\hat{h}_1 = 0 \\ c\hat{h}_2 + \mathrm{i}a_x U(z)\hat{h}_2 - \mathrm{D}\hat{u}_2 - \mathrm{i}a_x H(z)\hat{u}_2 - R_m^{-1}(\mathrm{D}^2 - a^2)\hat{h}_2 = 0 \\ c\hat{h}_3 + \mathrm{i}a_x U(z)\hat{h}_3 - \mathrm{D}\hat{u}_3 - \mathrm{i}a_x H(z)\hat{u}_3 - R_m^{-1}(\mathrm{D}^2 - a^2)\hat{h}_3 = 0 \end{cases} \tag{5.13}$$

with boundary conditions

$$\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3) = \mathbf{0}, \quad \hat{\mathbf{h}} = (\hat{h}_1, \hat{h}_2, \hat{h}_3) = \mathbf{0} \quad \text{at } z = \pm 1. \tag{5.14}$$

In deriving equations (5.13) we have considered general three-dimensional disturbances and we now show that the three-dimensional problem defined by (5.13) and (5.14) can be reduced to an equivalent two-dimensional one. To this purpose we introduce the Squire type transformations [17]

$$\begin{cases} \tilde{u} = a_x\hat{u}_1 + a_y\hat{u}_2, \quad \tilde{w} = a\hat{u}_3, \\ \tilde{h} = a_x\hat{h}_1 + a_y\hat{h}_2, \quad \tilde{k} = a\hat{h}_3, \\ a_x U(z) = a\tilde{U}(z), \quad a_x H(z) = a\tilde{H}(z), \\ \tilde{p} = a\hat{p}_1, \end{cases} \tag{5.15}$$

then equations (5.13) can be combined to give

$$\begin{cases} \big[\tilde{D}ac + 1 - \tilde{D}a\left(\mathrm{D}^2 - a^2\right)\big]\tilde{u} - \tilde{D}aP_m\big[\big(\mathrm{D} + \mathrm{i}a\tilde{H}(z)\big)\tilde{h} \\ \qquad\qquad + \tilde{H}'(z)\tilde{k}\big] = -\mathrm{i}a\tilde{p} \\ \big[\tilde{D}ac + 1 - \tilde{D}a\left(\mathrm{D}^2 - a^2\right)\big]\tilde{w} - \tilde{D}aP_m\big(\mathrm{D} + \mathrm{i}a\tilde{H}(z)\big)\tilde{k} = -\mathrm{D}\tilde{p} \\ \mathrm{D}\tilde{w} = -\mathrm{i}a\tilde{u} \\ \mathrm{D}\tilde{k} = -\mathrm{i}a\tilde{h} \\ \big[c + \mathrm{i}a\tilde{U}(z) - R_m^{-1}\big(\mathrm{D}^2 - a^2\big)\big]\tilde{h} + \tilde{H}'(z)\tilde{w} - \tilde{U}'(z)\tilde{k} \\ \qquad\qquad -\big[\mathrm{D} + \mathrm{i}a\tilde{H}(z)\big]\tilde{u} = 0 \\ \big[c + \mathrm{i}a\tilde{U}(z) - R_m^{-1}\big(\mathrm{D}^2 - a^2\big)\big]\tilde{k} - \big(\mathrm{D} + \mathrm{i}a\tilde{H}(z)\big)\tilde{w} = 0 \end{cases} \tag{5.16}$$

and the boundary conditions are

$$\tilde{u} = \tilde{w} = \tilde{h} = \tilde{k} = 0 \quad \text{at} \quad z = \pm 1.$$

These equations have exactly the same mathematical form as the original equations (5.13) with $a_y = u_2 = h_2 = 0$ and they define the equivalent two-dimensional problem. It is sufficient, therefore, to consider only two-dimensional disturbances; for, once the solution of equations (5.13) and (5.14) with $a_y = u_2 = h_2 = 0$ has been obtained, we can immediately obtain the corresponding solution of the equivalent two-dimensional problem by a trivial change in notation and from this, by means of transformations (5.15), we can then obtain the solution of the original three-dimensional problem. Therefore, in seeking sufficient criteria for linear stability of the basic laminar motion m_0, we may consider only two-dimensional perturbations $\mathbf{u}' = (u_1', 0, u_3')$, $\mathbf{h}' = (h_1', 0, h_3')$, and it is then convenient to introduce the stream functions ψ_1 and ψ_2 such that

$$u_1' = \frac{\partial \psi_1}{\partial z}, \quad u_3' = -\frac{\partial \psi_1}{\partial x},$$

$$h_1' = \frac{\partial \psi_2}{\partial z}, \quad h_3' = -\frac{\partial \psi_2}{\partial x}.$$

If we next let

$$\begin{cases} \psi_1(x, z, t) = \phi_1(z)\exp(\mathrm{i}ax + ct) \\ \psi_2(x, z, t) = \phi_2(z)\exp(\mathrm{i}ax + ct) \end{cases}$$

then

$$\begin{cases} \tilde{u} = \mathrm{D}\phi_1, \quad \tilde{w} = -\mathrm{i}a\phi_1, \\ \tilde{h} = \mathrm{D}\phi_2, \quad \tilde{k} = -\mathrm{i}a\phi_2, \end{cases}$$

and, substituting into (5.16), we have

$$\begin{cases} \left[\tilde{D}ac+1-\tilde{D}a(\mathrm{D}^2-a^2)\right]\mathrm{D}\phi_1-\tilde{D}aP_m\left[\mathrm{D}+\mathrm{i}aH(z)\right]\mathrm{D}\phi_2 \\ \qquad +\tilde{D}aP_m\mathrm{i}aH'(z)\phi_2=-\mathrm{i}a\tilde{p} \\ -\mathrm{i}a\left[\tilde{D}ac+1-\tilde{D}a(\mathrm{D}^2-a^2)\right]\phi_1 \\ \qquad +\mathrm{i}a\tilde{D}aP_m[\mathrm{D}+\mathrm{i}aH(z)]\phi_2=-\mathrm{D}\tilde{p} \\ \left[c+\mathrm{i}aU(z)-R_m^{-1}(\mathrm{D}^2-a^2)\right]\mathrm{D}\phi_2-\mathrm{i}aH'(z)\phi_1 \\ \qquad +\mathrm{i}aU'(z)\phi_2-\left[\mathrm{D}+\mathrm{i}aH(z)\right]\mathrm{D}\phi_1=0 \\ \left[c+\mathrm{i}aU(z)-R_m^{-1}\left(\mathrm{D}^2-a^2\right)\right]\phi_2-[\mathrm{D}+\mathrm{i}aH(z)]\phi_1=0, \end{cases} \tag{5.17}$$

under the boundary conditions

$$\phi_j(\pm 1)=\mathrm{D}\phi_j(\pm 1)=0, \quad j=1,2. \tag{5.18}$$

Eliminating the pressure $\tilde{p}$ between equations $(5.17)_1$, $(5.17)_2$ and applying the operator D^2-a^2 to equation $(5.17)_4$, we readily get

$$\begin{cases} \left[\tilde{D}ac+1-\tilde{D}a(\mathrm{D}^2-a^2)\right](\mathrm{D}^2-a^2)\phi_1 \\ \quad =\tilde{D}aP_m\left[(\mathrm{D}+\mathrm{i}aH(z))(\mathrm{D}^2-a^2)-\mathrm{i}aH''(z)\right]\phi_2 \\ \left\{\left[c-R_m^{-1}(\mathrm{D}^2-a^2)\right](\mathrm{D}^2-a^2)+\mathrm{i}a[U''(z)-a^2U(z)]\right. \\ \qquad \left.+2\mathrm{i}aU'(z)\mathrm{D}+\mathrm{i}aU(z)\mathrm{D}^2\right\}\phi_2=\left\{(\mathrm{D}^2-a^2)\mathrm{D}\right. \\ \qquad \left.+\mathrm{i}a[H''(z)-a^2H(z)]+2\mathrm{i}aH'(z)\mathrm{D}+\mathrm{i}aH(z)\mathrm{D}^2\right\}\phi_1. \end{cases} \tag{5.19}$$

Denoting by ϕ_j^* the complex conjugate of ϕ_j $(j=1,2)$, multiplying $(5.19)_1$ by ϕ_1^*, $(5.19)_2$ by $\tilde{D}aP_m\phi_2^*$, summing and integrating over the interval $(-1,1)$ we obtain the eigenvalue relation

$$\begin{aligned} &\left[(\tilde{D}ac+1)(I_1^2+a^2I_0^2)+\tilde{D}a(I_2^2+2a^2I_1^2+a^4I_0^2)\right] \\ &+\tilde{D}aP_m\left[c(J_1^2+a^2J_0^2)+R_m^{-1}(J_2^2+2a^2J_1^2+a^4J_0^2)\right]=\tilde{D}aP_m(Q+R), \end{aligned}$$

with

$$I_n^2=\int_{-1}^{1}|\mathrm{D}^n\phi_1|^2\mathrm{d}z, \quad J_n^2=\int_{-1}^{1}|\mathrm{D}^n\phi_2|^2\mathrm{d}z \quad (n=0,1,2),$$

$$\begin{aligned} Q=&-\int_{-1}^{1}\mathrm{D}^3\phi_2\phi_1^*\mathrm{d}z+a^2\int_{-1}^{1}\mathrm{D}\phi_2\phi_1^*\mathrm{d}z \\ &-2\mathrm{i}a\int_{-1}^{1}H'(z)\phi_2\mathrm{D}\phi_1^*\mathrm{d}z-\mathrm{i}a\int_{-1}^{1}H(z)\phi_2\mathrm{D}^2\phi_1^*\mathrm{d}z+\mathrm{i}a^3\int_{-1}^{1}H(z)\phi_2\phi_1^*\mathrm{d}z \end{aligned}$$

and

$$\begin{aligned}R &= \mathrm{i}a\int_{-1}^{1}\left[U''(z)-a^2U(z)\right]|\phi_2|^2\mathrm{d}z+2\mathrm{i}a\int_{-1}^{1}U'(z)\mathrm{D}\phi_2\phi_2^*\mathrm{d}z\\&-\int_{-1}^{1}\mathrm{D}^3\phi_1\phi_2^*\mathrm{d}z+\mathrm{i}a\int_{-1}^{1}U(z)\mathrm{D}^2\phi_2\phi_2^*\mathrm{d}z\\&-\mathrm{i}a\int_{-1}^{1}\left[H''(z)-a^2H(z)\right]\phi_1\phi_2^*\mathrm{d}z-2\mathrm{i}a\int_{-1}^{1}H'(z)\mathrm{D}\phi_1\phi_2^*\mathrm{d}z\\&+a^2\int_{-1}^{1}\mathrm{D}\phi_1\phi_2^*\mathrm{d}z-\mathrm{i}a\int_{-1}^{1}H(z)\mathrm{D}^2\phi_1\phi_2^*\mathrm{d}z.\end{aligned}$$

Thus

$$\begin{aligned}c_r=&\Big\{\tilde{D}aP_m\mathrm{Re}(Q+R)-\big[(I_1^2+a^2I_0^2)+\tilde{D}a(I_2^2+2a^2I_1^2+a^4I_0^2)\\&+\frac{\tilde{D}aP_m}{R_m}(J_2^2+2a^2J_1^2+a^4J_0^2)\big]\Big\}\Big\{\tilde{D}a(I_1^2+a^2I_0^2)+\tilde{D}aP_m(J_1^2+a^2J_0^2)\Big\}^{-1},\end{aligned}\tag{5.20}$$

$$c_i=\frac{P_m\mathrm{Im}(Q+R)}{I_1^2+a^2I_0^2+P_m(J_1^2+a^2J_0^2)},$$

with

$$\mathrm{Re}(Q+R)=\frac{\mathrm{i}a}{2}\int_{-1}^{1}U'(z)(\mathrm{D}\phi_2\phi_2^*-\mathrm{D}\phi_2^*\phi_2)\mathrm{d}z-\frac{\mathrm{i}a}{2}\int_{-1}^{1}H''(z)(\phi_1\phi_2^*-\phi_1^*\phi_2)\mathrm{d}z$$

and

$$\begin{aligned}\mathrm{Im}(Q+R) &= \int_{-1}^{1}(\phi_1\mathrm{D}^3\phi_2^*-\phi_1^*\mathrm{D}^3\phi_2)\mathrm{d}z\\&-a^2\int_{-1}^{1}(\phi_1\mathrm{D}\phi_2^*-\phi_1^*\mathrm{D}\phi_2)\mathrm{d}z-\mathrm{i}a\int_{-1}^{1}H(z)(\mathrm{D}\phi_1\phi_2^*+\mathrm{D}\phi_1^*\phi_2)\mathrm{d}z\\&+\mathrm{i}a\int_{-1}^{1}H(z)(\mathrm{D}\phi_1\mathrm{D}\phi_2^*+\mathrm{D}\phi_1^*\mathrm{D}\phi_2)\mathrm{d}z+\mathrm{i}a^3\int_{-1}^{1}H(z)(\phi_1\phi_2^*+\phi_1^*\phi_2)\mathrm{d}z\\&+\mathrm{i}a\int_{-1}^{1}[U''(z)-a^2U(z)]|\phi_2|^2\mathrm{d}z-\mathrm{i}a\int_{-1}^{1}U(z)|\mathrm{D}\phi_2|^2\mathrm{d}z\\&+\frac{\mathrm{i}a}{2}\int_{-1}^{1}U'(z)(\mathrm{D}\phi_2\phi_2^*+\mathrm{D}\phi_2^*\phi_2)\mathrm{d}z+\frac{\mathrm{i}a}{2}\int_{-1}^{1}H'(z)(\phi_1\mathrm{D}\phi_2^*+\phi_1^*\mathrm{D}\phi_2)\mathrm{d}z\\&+\frac{\mathrm{i}a}{2}\int_{-1}^{1}H'(z)(\mathrm{D}\phi_1\phi_2^*+\mathrm{D}\phi_1^*\phi_2)\mathrm{d}z.\end{aligned}$$

Setting

$$\mathcal{I}=\tilde{D}a\mathrm{Re}(Q+R)$$

and

$$\mathcal{D} = R_m(I_1^2 + a^2 I_0^2) + R_m\tilde{D}a(I_2^2 + 2a^2 I_1^2 + a^4 I_0^2) + \tilde{D}aP_m(J_2^2 + 2a^2 J_1^2 + a^4 J_0^2),$$

(5.20) becomes

$$c_r = \frac{1}{\tilde{D}a}\left(M^2\frac{\mathcal{I}}{\mathcal{D}} - 1\right)\mathcal{D}\left[I_1^2 + a^2 I_0^2 + P_m(J_1^2 + a^2 J_0^2)\right]^{-1} \tag{5.21}$$

with ϕ_1 and ϕ_2 belonging to the space of the kinematically admissible fields

$$\mathcal{H} = \left\{(\phi_1, \phi_2) \in \left(\mathrm{H}^2(-1,1)\right)^2 : \phi_i(\pm 1) = \mathrm{D}\phi_i(\pm 1) = 0 \quad \forall i = 1,2\right\}.$$

If we now define

$$\frac{1}{M_L^2(a)} = \max_{\mathcal{H}} \frac{\mathcal{I}}{\mathcal{D}}, \tag{5.22}$$

from (5.21) we readily deduce that the modes of wave number a are linearly stable for $M \leq M_L(a)$.

The Euler-Lagrange equations corresponding to the variational problem (5.22) are:

$$\begin{cases} R_m\left[-1 + \tilde{D}a(\mathrm{D}^2 - a^2)\right](\mathrm{D}^2 - a^2)\phi_1 + \mathrm{i}a\sigma H''\phi_2 = 0 \\ \tilde{D}aP_m(\mathrm{D}^2 - a^2)^2\phi_2 + \mathrm{i}a\sigma\left(-H''\phi_1 + 2U'\mathrm{D}\phi_2 + U''\phi_2\right) = 0, \end{cases} \tag{5.23}$$

with boundary conditions (5.18). $M_L^2(a)$ is the least positive eigenvalue of the characteristic value problem (5.23) with boundary conditions (5.18).

Finally, if

$$M \leq M_c = \min_{a>0} M_L(a), \tag{5.24}$$

then all modes are stable and hence the basic laminar motion m_0 is linearly stable.

5.4 Sufficient condition for global non linear exponential stability

In order to study the nonlinear stability of m_0 by employing the energy method we introduce the Liapunov function

$$E(t) = \frac{1}{2}(\tilde{D}a\|\mathbf{u}\|^2 + \tilde{D}aP_m\|\mathbf{h}\|^2)$$

where $\|\cdot\|$ denotes as usual the $\mathrm{L}^2(\Omega)$ norm and

$$\Omega = \left[0, \frac{2\pi}{a_x}\right] \times \left[0, \frac{2\pi}{a_y}\right] \times [-1, 1]$$

is the period cell. Taking into account the periodicity and the boundary conditions, we find along the solutions of (5.9)

$$\dot{E}(t) = \frac{1}{R_m}\left[M^2\frac{\mathscr{I}(\mathbf{u},\mathbf{h})}{\mathscr{D}(\mathbf{u},\mathbf{h})} - 1\right]\mathscr{D}(\mathbf{u},\mathbf{h}), \tag{5.25}$$

where

$$\mathscr{I}(\mathbf{u},\mathbf{h}) = \tilde{D}a\left[\int_\Omega U'(z)h_1h_3\mathrm{d}\Omega + \int_\Omega H'(z)(h_3u_1 - h_1u_3)\mathrm{d}\Omega\right],$$

$$\mathscr{D}(\mathbf{u},\mathbf{h}) = R_m\left(\|\mathbf{u}\|^2 + \tilde{D}a\|\nabla\mathbf{u}\|^2\right) + \tilde{D}aP_m\|\nabla\mathbf{h}\|^2,$$

and the perturbations $\mathbf{u}$, $\mathbf{h}$ belong to the kinematically admissible space

$$\mathscr{W} = \{(\mathbf{u},\mathbf{h}) \in (\mathrm{H}^1(\Omega))^6 : \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{h} = 0,\ \mathbf{u}(x,y,\pm1) = \mathbf{h}(x,y,\pm1) = 0\}.$$

Let now

$$\frac{1}{M_E^2} = \max_{\mathscr{W}}\frac{\mathscr{I}}{\mathscr{D}}. \tag{5.26}$$

The existence of the maximum of the functional $\mathscr{I}/\mathscr{D}$ can proved by following the proof of Theorem 4.2.

By assuming $M < M_E$, from (5.25) and since, by Poincaré inequality,

$$\mathscr{D}(\mathbf{u},\mathbf{h}) \geq \frac{1}{4}\left[R_m\left(4 + \tilde{D}a\pi^2\right)\|\mathbf{u}\|^2 + \tilde{D}aP_m\pi^2\|\mathbf{h}\|^2\right],$$

we obtain the energy inequality

$$\dot{E}(t) \leq \frac{1}{R_m}\left(\frac{M^2}{M_E^2} - 1\right)\nu_0E(t), \tag{5.27}$$

where

$$\nu_0 = \frac{1}{2}\min\left\{\frac{R_m(4 + \tilde{D}a\pi^2)}{\tilde{D}a}, \pi^2\right\}. \tag{5.28}$$

Integrating (5.27) we have global nonlinear exponential stability of the basic motion m_0 according to the following inequality

$$E(t) \leq E(0)\exp\left[\frac{1}{R_m}\left(\frac{M^2}{M_E^2} - 1\right)\nu_0t\right].$$

The Euler-Lagrange equations corresponding to the variational problem (5.26) are

$$\begin{cases} \lambda\left[\tilde{D}a\,(H'h_3\mathbf{i} - H'h_1\mathbf{k})\right] + R_m(\mathbf{u} - \tilde{D}a\Delta\mathbf{u}) = -\nabla p' \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{h} = 0 \\ \lambda\,(H'u_1\mathbf{k} - H'u_3\mathbf{i} + U'h_3\mathbf{i} + U'h_1\mathbf{k}) - P_m\Delta\mathbf{h} = -\nabla\chi \end{cases} \tag{5.29}$$

where p', χ are Lagrange multipliers associated with the divergence constraints. This system must be solved subject to the boundary conditions (5.10) and M_E^2 is then obtained as the least positive eigenvalue λ of the characteristic-value problem (5.29) and (5.10).

Remark 5.1 (An estimate of M_E)**.** By Cauchy and Poincarè inequalities we have

$$\begin{aligned}\frac{\mathscr{I}(\mathbf{u},\mathbf{h})}{\mathscr{D}(\mathbf{u},\mathbf{h})} &\le 4\tilde{D}a\frac{q\|\mathbf{u}\|^2+(r+q)\|\mathbf{h}\|^2}{R_m(4+\tilde{D}a\pi^2)\|\mathbf{u}\|^2+\tilde{D}aP_m\pi^2\|\mathbf{h}\|^2} \\ &\le 4\max\left\{\frac{q\tilde{D}a}{R_m(4+\tilde{D}a\pi^2)},\frac{r+q}{P_m\pi^2}\right\},\end{aligned} \tag{5.30}$$

where

$$2r=\max_{z\in[-1,1]}|U'(z)|,\quad 2q=\max_{z\in[-1,1]}|H'(z)|.$$

Consequently, from (5.26) and (5.30) we deduce that

$$M_E \ge M_E^* = \frac{1}{2}\sqrt{\min\left\{\frac{R_m(4+\tilde{D}a\pi^2)}{q\tilde{D}a},\frac{P_m\pi^2}{r+q}\right\}},$$

and therefore the condition $M < M_E^*$ implies that m_0 is globally nonlinearly exponentially stable.

Let us consider now the set of two-dimensional disturbances in the xz-plane

$$\mathscr{V}=\left\{(\mathbf{u},\mathbf{h})\in\mathscr{W}:\mathbf{u}=\frac{\partial\psi_1}{\partial z}\mathbf{i}-\frac{\partial\psi_1}{\partial x}\mathbf{k},\ \mathbf{h}=\frac{\partial\psi_2}{\partial z}\mathbf{i}-\frac{\partial\psi_2}{\partial x}\mathbf{k}\right\}.$$

Since $\mathscr{V}$ is a closed subset of $\mathscr{W}$

$$\exists\max_{\mathscr{V}}\frac{\mathscr{I}}{\mathscr{D}}=\frac{1}{M_E'^2}\le\frac{1}{M_E^2}.$$

Therefore $M_E \le M_E'$ and hence two-dimensional disturbances in the xz-plane are more stable than three-dimensional ones. For such disturbances the Euler-Lagrange equations (5.29) reduce to

$$\begin{cases} R_m\left(-\Delta\psi_1+\tilde{D}a\Delta^2\psi_1\right)+\lambda\tilde{D}aH''\dfrac{\partial\psi_2}{\partial x}=0 \\ P_m\Delta^2\psi_2+\lambda\left(-H''\dfrac{\partial\psi_1}{\partial x}+2U'\dfrac{\partial^2\psi_2}{\partial x\partial z}+U''\dfrac{\partial\psi_2}{\partial x}\right)=0\end{cases}$$

whose normal mode form is identical with equations (5.23) and thus, for disturbances of this type, the linear theory and the energy method lead to identical results.

5.4.1 Non linear stability of the MHD laminar flows with respect to disturbances normal to the embedding magnetic field

Let $\mathcal{S}$ be the set of the two-dimensional perturbations considered by Rionero and Maiellaro in [74]

$$\begin{cases} \mathbf{u}(z,t) = u_1(z,t)\mathbf{i} + u_2(z,t)\mathbf{j} \\ \mathbf{h}(z,t) = h_1(z,t)\mathbf{i} + h_2(z,t)\mathbf{j}. \end{cases} \tag{5.31}$$

Introducing the energy

$$\mathcal{E}(t) = \frac{\tilde{D}a}{2}\int_{-1}^{1} |\mathbf{u}(z,t)|^2 \mathrm{d}z + \frac{\tilde{D}aP_m}{2}\int_{-1}^{1} |\mathbf{h}(z,t)|^2 \mathrm{d}z,$$

the following energy inequality is easily obtained

$$\dot{\mathcal{E}}(t) \leq -\frac{\nu_0}{R_m}\mathcal{E}(t), \tag{5.32}$$

with ν_0 defined through (5.28).

Integrating (5.32) one can prove that m_0 is globally nonlinearly exponentially stable with respect to laminar disturbances (5.31) for all Hartmann numbers according to

$$\mathcal{E}(t) \leq \mathcal{E}(0)\exp\left(-\frac{\nu_0}{R_m}t\right).$$

5.5 The convergence of the Galerkin method

In using the Galerkin method to solve the eigenvalue problem (5.23) with boundary conditions (5.18), we expand the eigenfunctions (ϕ_1, ϕ_2) in terms of the set $\{C_n, S_n\}_{n\in\mathbb{N}}$ introduced by Harris and Reid in [26] which is orthonormal with respect to the $\mathrm{L}^2[-1,1]$ inner product

$$(f,g) = \int_{-1}^{1} f(z)g(z)\mathrm{d}z$$

and complete in the space $\left\{f \in \mathrm{H}^2[-1,1]/f(\pm 1) = \mathrm{D}f(\pm 1) = 0\right\}$ with the second derivative norm.

$$C_n(z) = \frac{1}{\sqrt{2}}\left[\frac{\cosh(\lambda_n z)}{\cosh\lambda_n} - \frac{\cos(\lambda_n z)}{\cos\lambda_n}\right]$$

and

$$S_n(z) = \frac{1}{\sqrt{2}}\left[\frac{\sinh(\mu_n z)}{\sinh\mu_n} - \frac{\sin(\mu_n z)}{\sin\mu_n}\right],$$

λ_n and μ_n being roots of the characteristic equations

$$\tanh\lambda + \tan\lambda = 0 \quad \text{and} \quad \coth\mu - \cot\mu = 0, \tag{5.33}$$

respectively, are solutions of the Sturm-Liouville problem

$$\mathrm{D}^4 y - \nu^4 y = 0$$

with boundary conditions $y(\pm 1) = \mathrm{D}y(\pm 1) = 0$. $2\lambda_n$ and $2\mu_n$ are listed in [11], page 636[2].

We now expand ϕ_1 and ϕ_2 in terms of $\{C_n, S_n\}_{n\in\mathbb{N}}$

$$\begin{cases} \phi_1 = \sum_{n=1}^{+\infty} \left[A_n^{(C)} C_n(z) + A_n^{(S)} S_n(z)\right], \\ \phi_2 = \sum_{n=1}^{+\infty} \left[B_n^{(C)} C_n(z) + B_n^{(S)} S_n(z)\right], \end{cases} \tag{5.34}$$

insert these series into system (5.23) and by requiring that the error in the differential equations (5.23) be orthogonal to C_r and S_r for each positive integer r, we obtain an infinite system of linear homogeneous equations for the constants $A_n^{(C)}$, $A_n^{(S)}$, $B_n^{(C)}$ and $B_n^{(S)}$. In order that these constants do not vanish identically, the determinant of the system must vanish, and this condition yields the secular determinant

$$\begin{vmatrix} Z_{nr} & 0 & \mathrm{i}a\sigma E_{nr}^{(C)} & \mathrm{i}a\sigma E_{nr}^{(S)} \\ 0 & F_{nr} & \mathrm{i}a\sigma G_{nr}^{(C)} & \mathrm{i}a\sigma G_{nr}^{(S)} \\ -\mathrm{i}a\sigma E_{nr}^{(C)} & -\mathrm{i}a\sigma E_{nr}^{(S)} & L_{nr} + \mathrm{i}a\sigma X_{nr}^{(C)} & \mathrm{i}a\sigma X_{nr}^{(S)} \\ -\mathrm{i}a\sigma G_{nr}^{(C)} & -\mathrm{i}a\sigma G_{nr}^{(S)} & \mathrm{i}a\sigma Y_{nr}^{(C)} & Q_{nr} + \mathrm{i}a\sigma Y_{nr}^{(S)} \end{vmatrix} = 0 \tag{5.35}$$

where

$$Z_{nr} = R_m \left\{ [a^2 + \tilde{D}a(\lambda_n^4 + a^4)]\delta_{nr} - (1 + 2a^2\tilde{D}a) \int_{-1}^{1} C_n''(z) C_r(z)\mathrm{d}z \right\},$$

$$F_{nr} = R_m \left\{ [a^2 + \tilde{D}a(\mu_n^4 + a^4)]\delta_{nr} - (1 + 2a^2\tilde{D}a) \int_{-1}^{1} S_n''(z) S_r(z)\mathrm{d}z \right\},$$

$$E_{nr}^{(C)} = \int_{-1}^{1} H''(z) C_n(z) C_r(z)\mathrm{d}z, \quad E_{nr}^{(S)} = \int_{-1}^{1} H''(z) S_n(z) C_r(z)\mathrm{d}z,$$

$$G_{nr}^{(C)} = E_{rn}^{(S)}, \quad G_{nr}^{(S)} = \int_{-1}^{1} H''(z) S_n(z) S_r(z)\mathrm{d}z,$$

$$L_{nr} = \tilde{D}aP_m \left[(\lambda_n^4 + a^4)\delta_{nr} - 2a^2 \int_{-1}^{1} C_n''(z) C_r(z)\mathrm{d}z \right],$$

[2] In [11] the roots of the characteristic equations (5.33) are derived for the interval $[-1/2, 1/2]$ and here we have re-calculated them for the interval [-1,1].

$$Q_{nr} = \tilde{D}aP_m\left[(\mu_n^4 + a^4)\delta_{nr} - 2a^2\int_{-1}^{1} S_n''(z)S_r(z)\mathrm{d}z\right],$$

$$X_{nr}^{(C)} = \int_{-1}^{1} U'(z)\left[C_n'(z)C_r(z) - C_n(z)C_r'(z)\right]\mathrm{d}z,$$

$$X_{nr}^{(S)} = \int_{-1}^{1} U'(z)\left[S_n'(z)C_r(z) - S_n(z)C_r'(z)\right]\mathrm{d}z,$$

$$Y_{nr}^{(C)} = -X_{rn}^{(S)} \text{ and } Y_{nr}^{(S)} = \int_{-1}^{1} U'(z)\left[S_n'(z)S_r(z) - S_n(z)S_r'(z)\right]\mathrm{d}z.$$

By considering only a finite number of terms (say N) in the expansions (5.34), (5.35) reduces to an algebraic equation in the unknown σ whose least positive solution is the so-called N-*th* approximation of $M_L^2(a)$. With $N = 1, 2, ...$, we get a non-increasing sequence of approximations of the critical Hartmann number which, as we shall prove, converges to the exact value of $M_L(a)$. The convergence of the Galerkin method is based on the following Mikhlin Theorem [14, 49].

Theorem 5.1 (Mikhlin). *Let λ be a parameter in the equation*

$$\mathcal{A}u - \lambda\mathcal{K}u = 0, \tag{5.36}$$

where $\mathcal{A}$ and $\mathcal{K}$ are linear operators, and the domain of $\mathcal{A}$, $D_{\mathcal{A}}$, is a linear space which is dense in a Hilbert space $\mathcal{H}$ with inner product $< \cdot, \cdot >$. Let $D_{\mathcal{A}}$ be contained in the domain of $\mathcal{K}$, $D_{\mathcal{K}}$, and assume that

1) *$\mathcal{A}$ is a positive-definite self-adjoint operator,*

2) *the operator $\mathcal{A}^{-1}\mathcal{K}$ can be extended to be completely continuous on the Hilbert space $\mathcal{H}_0$ which is the completion of $D_{\mathcal{A}}$ under the norm $< \mathcal{A}\cdot, \cdot >^{1/2}$.*

Then the Galerkin method for calculating the eigenvalues of (5.36) *is a convergent process in $\mathcal{H}_0$.*

The eigenvalue problem (5.23) with boundary conditions (5.18) can be cast in a form such that the conditions of Mikhlin Theorem are satisfied. In fact, by setting $\lambda = -\mathrm{i}a\sigma$,

$$\mathcal{A} = \begin{pmatrix} R_m\left[-1 + \tilde{D}a(\mathrm{D}^2 - a^2)\right](\mathrm{D}^2 - a^2) & 0 \\ 0 & \tilde{D}aP_m(\mathrm{D}^2 - a^2)^2 \end{pmatrix} \tag{5.37}$$

and

$$\mathcal{K} = \begin{pmatrix} 0 & H''(z) \\ -H''(z) & 2U'(z)\mathrm{D} + U''(z) \end{pmatrix}, \tag{5.38}$$

equations (5.23) can be rewritten as

$$\mathcal{A}\mathbf{u} = \lambda\mathcal{K}\mathbf{u}$$

where $\mathbf{u}$ is the column vector with components u_1 and u_2.

The domain of the linear operator $\mathcal{A}$ is the vector space

$$D_{\mathcal{A}} = \left\{\mathbf{u} = (u_1, u_2) \in \left(\mathrm{C}^4([-1,1],\mathbb{C})\right)^2 : u_j(\pm 1) = \mathrm{D}u_j(\pm 1) = 0 \ \forall j = 1,2\right\}$$

which is dense in the Hilbert space $\left(\mathrm{L}^2([-1,1],\mathbb{C})\right)^2$ with inner product

$$(\mathbf{u},\mathbf{v}) = \int_{-1}^{1} (u_1 v_1^* + u_2 v_2^*)\,\mathrm{d}z$$

and norm

$$\|\mathbf{u}\| = \left[\int_{-1}^{1} (|u_1|^2 + |u_2|^2)\mathrm{d}z\right]^{1/2}.$$

It can be readily established by direct integration by parts that $\mathcal{A}$ is positive-definite on the space $D_{\mathcal{A}}$. In fact

$$\begin{aligned}(\mathcal{A}\mathbf{u},\mathbf{u}) =& R_m \int_{-1}^{1} [-1 + \tilde{D}a(\mathrm{D}^2 - a^2)](\mathrm{D}^2 - a^2)u_1 u_1^* \mathrm{d}z \\ &+ \tilde{D}aP_m \int_{-1}^{1} (\mathrm{D}^2 - a^2)^2 u_2 u_2^* \mathrm{d}z \\ &= R_m \int_{-1}^{1} [|\mathrm{D}u_1|^2 + a^2|u_1|^2 + \tilde{D}a|(\mathrm{D}^2 - a^2)u_1|^2]\mathrm{d}z \\ &+ \tilde{D}aP_m \int_{-1}^{1} |(\mathrm{D}^2 - a^2)u_2|^2 \mathrm{d}z \geq 0 \quad \forall \mathbf{u} \in D_{\mathcal{A}}\end{aligned}$$

and the equality holds if and only if $\mathbf{u} = \mathbf{0}$. Similarly we can prove that

$$(\mathcal{A}\mathbf{u},\mathbf{v}) = (\mathbf{u},\mathcal{A}\mathbf{v}) \quad \forall \mathbf{u},\mathbf{v} \in D_{\mathcal{A}},$$

viz $\mathcal{A}$ is self-adjoint on $D_{\mathcal{A}}$.

It is easy to check that $(\mathcal{A}\mathbf{u},\mathbf{v})$ defines an inner product on $D_{\mathcal{A}}$ with the corresponding norm

$$\begin{aligned}\|\mathbf{u}\|_{\mathcal{A}}^2 := (\mathcal{A}\mathbf{u},\mathbf{u}) =& R_m \int_{-1}^{1} [|\mathrm{D}u_1|^2 + a^2|u_1|^2 + \tilde{D}a|(\mathrm{D}^2 - a^2)u_1|^2]\mathrm{d}z \\ &+ \tilde{D}aP_m \int_{-1}^{1} |(\mathrm{D}^2 - a^2)u_2|^2 \mathrm{d}z,\end{aligned}$$

and let $\mathcal{H}_0$ be the completion of $D_{\mathcal{A}}$ under the norm $\|\cdot\|_{\mathcal{A}}$.
Let us consider now the Hilbert space of the kinematically admissible fields introduced in section 5.3

$$\mathcal{H} = \left\{\mathbf{u} = (u_1, u_2) \in (\mathrm{H}^2([-1,1],\mathbb{C})^2 : u_j(\pm 1) = \mathrm{D}u_j(\pm 1) = 0 \ \forall j = 1,2\right\}$$

and observe that it is the completion of $D_{\mathcal{A}}$ under the norm

$$\|\mathbf{u}\|_{\mathcal{H}} = \left(\sum_{j=1}^{2} \|\mathrm{D}^2 u_j\|_2^2\right)^{\frac{1}{2}},$$

$\|\cdot\|_2$ denoting the standard $\mathrm{L}^2([-1,1],\mathbb{C})$-norm. The well-known isoperimetric inequalities

$$\|\mathrm{D}^2 w\|_2^2 \geq \lambda_1^4 \|w\|_2^2 \quad \forall w \in \mathrm{H}^2([-1,1],\mathbb{C}),\ w(\pm 1) = \mathrm{D}w(\pm 1) = 0, \tag{5.39}$$

and

$$\|\mathrm{D}w\|_2^2 \geq \frac{\pi^2}{4}\|w\|_2^2 \quad \forall w \in \mathrm{H}^1([-1,1],\mathbb{C}),\ w(\pm 1) = 0, \tag{5.40}$$

with λ_1 the least positive root of the characteristic equation $(5.33)_1$, give

$$\sqrt{\tilde{D}a \min\{R_m, P_m\}}\|\mathbf{u}\|_{\mathcal{H}} \leq \|\mathbf{u}\|_{\mathcal{A}}$$
$$\leq \sqrt{R_m\left(\frac{4}{\pi^2} + \frac{a^2}{\lambda_1^4}\right) + \tilde{D}a \max\{R_m, P_m\}\left(1 + \frac{8a^2}{\pi^2} + \frac{a^4}{\lambda_1^4}\right)}\|\mathbf{u}\|_{\mathcal{H}},$$

for all $\mathbf{u} \in D_{\mathcal{A}}$, viz $\|\cdot\|_{\mathcal{A}}$ and $\|\cdot\|_{\mathcal{H}}$ are equivalent norms on the space $D_{\mathcal{A}}$ and hence $\mathcal{H}_0 = \mathcal{H}$.

Concerning the linear operator $\mathcal{K}$ defined in (5.38), since U and H are smooth functions in $[-1,1]$, the domain of $\mathcal{K}$ is the Banach space

$$D_{\mathcal{K}} = \mathrm{L}^2([-1,1],\mathbb{C}) \times \mathrm{H}^1([-1,1],\mathbb{C})$$

endowed with the norm $\|\mathbf{u}\|_{D_{\mathcal{K}}} = \|u_1\|_2 + \|u_2\|_2 + \|\mathrm{D}u_2\|_2$, and it can be readily shown that $\mathcal{K}$ is bounded. Obviously $D_{\mathcal{A}}$ is contained in $D_{\mathcal{K}}$.

Let now $\mathcal{T} = \mathcal{A}^{-1}\mathcal{K}$:

$$\mathcal{T} : \mathbf{u} \in D_{\mathcal{K}} \mapsto \mathcal{T}\mathbf{u} = \int_{-1}^{1} \mathbf{G}(z,\xi)\mathcal{K}(\xi)\mathbf{u}(\xi)\mathrm{d}\xi, \tag{5.41}$$

where $\mathbf{G}$ is the matrix Green function corresponding to the matrix differential operator $\mathcal{A}$ with boundary conditions (5.18) (see Appendix A for details). Since the elements G_{11} and G_{22} in the matrix Green function $\mathbf{G}$ belong to $\mathrm{C}^2([-1,1] \times [-1,1])$ and vanish with their first derivatives with respect to z at the boundaries $z = \pm 1$, the range of $\mathcal{T}$ is contained in the Banach space

$$\mathscr{H} = \left\{ \mathbf{u} = (u_1, u_2) \in \left(\mathrm{C}^2([-1,1],\mathbb{C})\right)^2 : u_j(\pm 1) = \mathrm{D}u_j(\pm 1) = 0 \, \forall j = 1, 2 \right\}$$

endowed with the standard norm

$$\|\mathbf{u}\|_{\mathscr{H}} = \sum_{h,j=1}^{2} \max_{z\in[-1,1]} |\mathrm{D}^h u_j(z)|.$$

Now we prove that $\mathcal{T}$ is completely continuous, i.e. $\mathcal{T}$ takes bounded sequences in $D_{\mathcal{K}}$ into sequences in $\mathscr{H}$ with a convergent subsequence. Let $\{\mathbf{u}_n = (u_{1,n}, u_{2,n})\}_{n\in\mathbb{N}}$ be a bounded sequence in $D_{\mathcal{K}}$ and $\epsilon > 0$, then there exist c, l, $\delta_\epsilon \in \mathbb{R}^+$ such that

$$\|\mathbf{u}_n\|_{D_{\mathcal{K}}} = \|u_{1,n}\|_2 + \|u_{2,n}\|_2 + \|\mathrm{D}u_{2,n}\|_2 \le c \ \forall n \in \mathbb{N},$$

$$\left| \frac{\partial^h G_{jj}}{\partial z^h}(z,\xi) \right| \le l \quad \forall j = 1,2, \ \forall h = 0,1,2$$

and

$$\left| \frac{\partial^h G_{jj}}{\partial z^h}(z',\xi) - \frac{\partial^h G_{jj}}{\partial z^h}(z,\xi) \right| < \epsilon \ \forall j = 1,2, \ \forall |z'-z| < \delta_\epsilon, \ \forall h = 0,1,2.$$

Then, setting $\mathcal{T}\mathbf{u}_n = ((\mathcal{T}\mathbf{u}_n)_1, (\mathcal{T}\mathbf{u}_n)_2)$, we have

$$|\mathrm{D}^h(\mathcal{T}\mathbf{u}_n)_j(z)| \le lc \sup_{\|\mathbf{w}\|_{D_{\mathcal{K}}}=1} \|\mathcal{K}\mathbf{w}\| \quad \forall j = 1,2, \ \forall z \in [-1,1],$$
$$\forall h = 0,1,2, \ \forall n \in \mathbb{N}$$

and

$$|\mathrm{D}^h(\mathcal{T}\mathbf{u}_n)_j(z') - \mathrm{D}^h(\mathcal{T}\mathbf{u}_n)_j(z)| \le \epsilon c \sup_{\|\mathbf{w}\|_{D_{\mathcal{K}}}=1} \|\mathcal{K}\mathbf{w}\| \quad \forall j = 1,2,$$
$$\forall |z'-z| < \delta_\epsilon, \ \forall h = 0,1,2, \ \forall n \in \mathbb{N}.$$

Therefore $\{\mathrm{D}^h(\mathcal{T}\mathbf{u}_n)_j\}_{n\in\mathbb{N}}$ is a uniformly bounded, equicontinuous sequence for all $j = 1,2$ and $h = 0,1,2$, and so, by Ascoli-Arzelà Theorem, there exists a subsequence $\{\mathcal{T}\mathbf{u}_{k_n}\}_{n\in\mathbb{N}} \subset \{\mathcal{T}\mathbf{u}_n\}_{n\in\mathbb{N}}$ such that $\{\mathrm{D}^h(\mathcal{T}\mathbf{u}_{k_n})_j\}_{n\in\mathbb{N}}$ converges uniformly in $[-1,1]$ for all $j = 1,2$ and $h = 0,1,2$. Then $\{\mathcal{T}\mathbf{u}_{k_n}\}_{n\in\mathbb{N}}$ converges in $\mathscr{H}$.
Finally, since the Hilbert space $\mathcal{H}$ is contained in $D_{\mathcal{K}}$ with[3]

$$\|\mathbf{u}\|_{\mathcal{H}} \ge \frac{\pi}{4} \|\mathbf{u}\|_{D_{\mathcal{K}}} \quad \forall \mathbf{u} \in \mathcal{H}, \tag{5.42}$$

[3]The inequality (5.42) is obtained combining the inequalities (5.39) and (5.40).

and since the uniform convergence in $[-1, 1]$ implies the convergence in $\mathrm{L}^2[-1, 1]$-norm we deduce that $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is completely continuous. Therefore the conditions of Mikhlin Theorem are established and thus the Galerkin method for computing the eigenvalues of the characteristic-value problem (5.23)-(5.18) is a convergent process in the space of the kinematically admissible fields $\mathcal{H}$. In particular the sequence of the approximations of the critical Hartmann number M_c defined in (5.24) converges to the exact value.

Appendix A

Matrix Green function

The matrix Green function $\mathbf{G}(z,\xi)$ introduced in (5.41) is given by

$$\mathbf{G}(z,\xi) = \begin{pmatrix} G_{11}(z,\xi) & 0 \\ 0 & G_{22}(z,\xi) \end{pmatrix}$$

where

$$G_{11}(z,\xi) = \begin{cases} g_{11}(z,\xi) & \text{if } z \le \xi \\ g_{11}(z,\xi) + \dfrac{a\sinh\alpha(z-\xi) - \alpha\sinh a(z-\xi)}{a\alpha\tilde{D}aR_m(\alpha^2-a^2)} & \text{if } z > \xi, \end{cases}$$

$$G_{22}(z,\xi) = \begin{cases} g_{22}(z,\xi) & \text{if} z \le \xi \\ g_{22}(z,\xi) + \dfrac{a(z-\xi)\cosh a(z-\xi) - \sinh a(z-\xi)}{2\tilde{D}aP_m a^3} & \text{if} z > \xi, \end{cases}$$

with $\alpha = \sqrt{\dfrac{1+a^2\tilde{D}a}{\tilde{D}a}}$,

$$\begin{aligned}
g_{11}(z,\xi) = {} & \frac{\alpha\sinh a(1-\xi) - a\sinh\alpha(1-\xi)}{a\alpha\tilde{D}aR_m(\alpha^2-a^2)} \\
& \cdot\left\{\frac{a\alpha[\cosh a(1-z) - \cosh 2\alpha\cosh a(z+1)] + \alpha^2\sinh 2\alpha\sinh a(z+1)}{2a\alpha(1-\cosh 2\alpha\cosh 2a) + (a^2+\alpha^2)\sinh 2\alpha\sinh 2a}\right. \\
& + \left.\frac{a\alpha[\cosh\alpha(1-z) - \cosh 2a\cosh\alpha(z+1)] + a^2\sinh 2a\sinh\alpha(z+1)}{2a\alpha(1-\cosh 2\alpha\cosh 2a) + (a^2+\alpha^2)\sinh 2\alpha\sinh 2a}\right\} \\
& + \frac{\cosh a(1-\xi) - \cosh\alpha(1-\xi)}{\tilde{D}aR_m(\alpha^2-a^2)} \\
& \cdot\left\{\frac{a\sinh 2\alpha\cosh a(z+1) - \alpha[\sinh a(1-z) + \cosh 2\alpha\sinh a(z+1)]}{2a\alpha(1-\cosh 2\alpha\cosh 2a) + (a^2+\alpha^2)\sinh 2\alpha\sinh 2a}\right. \\
& + \left.\frac{\alpha\sinh 2a\cosh\alpha(z+1) - a[\sinh\alpha(1-z) + \cosh 2a\sinh\alpha(z+1)]}{2a\alpha(1-\cosh 2\alpha\cosh 2a) + (a^2+\alpha^2)\sinh 2\alpha\sinh 2a}\right\}
\end{aligned}$$

and

$$
\begin{aligned}
g_{22}(z,\xi) = &\frac{\sinh a(1-\xi) - a(1-\xi)\cosh a(1-\xi)}{2\tilde{D}aP_m a^3(4a^2-\sinh^2 2a)}\big\{a^2\cosh a(1-z) \\
&- [(a^2+1)\sinh 2a + 2a\cosh 2a]\sinh a(z+1) \\
&+ a(\sinh 2a + a\cosh 2a)\cosh a(z+1)\big\} \\
&+ \frac{(1-\xi)\sinh a(1-\xi)}{2\tilde{D}aP_m a(4a^2-\sinh^2 2a)}\big\{a\sinh a(1-z) \\
&- (\sinh 2a + a\cosh 2a)\sinh a(z+1) + a\sinh 2a\cosh a(z+1)\big\} \\
&+ z\frac{\sinh a(1-\xi) - a(1-\xi)\cosh a(1-\xi)}{2\tilde{D}aP_m a^2(4a^2-\sinh^2 2a)}\big\{a\cosh a(1-z) \\
&+ \sinh a(1-z) \\
&+ (\cosh 2a - a\sinh 2a)\sinh a(z+1) + a\cosh 2a\cosh a(z+1)\big\} \\
&+ z\frac{(1-\xi)\sinh a(1-\xi)}{2\tilde{D}aP_m a(4a^2-\sinh^2 2a)}\big\{a\sinh a(1-z) \\
&+ (\sinh 2a - a\cosh 2a)\sinh a(z+1) + a\sinh 2a\cosh a(z+1)\big\}.
\end{aligned}
$$

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